

Mid-Cities Math Circle (MC)²
Factoring
April 12, 2023

Warm-up problems

Problem 1. Compute $100003^2 - 99997^2$.

Problem 2. Evaluate

$$\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{2+\sqrt{3}} + \frac{1}{\sqrt{5}+2} + \frac{1}{\sqrt{6}+\sqrt{5}} + \frac{1}{\sqrt{7}+\sqrt{6}} + \frac{1}{2\sqrt{2}+\sqrt{7}} + \frac{1}{3+2\sqrt{2}}.$$

Problem 3. Evaluate

$$\frac{1}{\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{4}} + \frac{1}{\sqrt[3]{4} + \sqrt[3]{6} + \sqrt[3]{9}} + \frac{1}{\sqrt[3]{9} + \sqrt[3]{12} + \sqrt[3]{16}} + \frac{1}{\sqrt[3]{16} + \sqrt[3]{20} + \sqrt[3]{25}}.$$

Harder problems

Problem 4. Compute

$$2\sqrt{\frac{3}{2} + \sqrt{2}} - \left(\frac{3}{2} + \sqrt{2}\right).$$

Problem 5. The number 27000001 has exactly four prime factors. Find their sum.

Problem 6. Find the largest divisor of 1001001001 that does not exceed 10000.

Problem 7. Find all nonnegative integers m such that the number $m^5 + m + 1$ is prime.

Problem 8. Evaluate the product

$$(\sqrt{5} + \sqrt{6} + \sqrt{7}) (\sqrt{5} + \sqrt{6} - \sqrt{7}) (\sqrt{5} - \sqrt{6} + \sqrt{7}) (-\sqrt{5} + \sqrt{6} + \sqrt{7}).$$

Problem 9. (i) Prove that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc).$$

(ii) Compute $\frac{(2023)^3 - (1016)^3 - (1007)^3}{2023 \cdot 1016 \cdot 1007}$.

Problem 10. Determine the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3.$$

Problem 11. Let a, b, c be the roots of $x^3 + 5x - 2013 = 0$. Find $a^3 + b^3 + c^3$.

Problem 12. Let x, y , and z be positive numbers satisfying

$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad \text{and} \quad z + \frac{1}{x} = \frac{7}{3}$$

What is the value of xyz ?

Problem 13. Find all of the 6 different prime factors of 512064008001 given that 13 is one of them.