UT Arlington Mid-Cities Math Circle $(MC)^2$
Symmetries in Geometry
March 8, 2023

Warm-up Problems

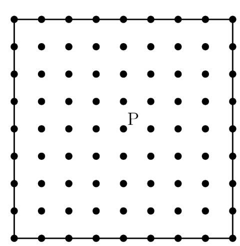
Problem 1. Is there a bounded plane figure, which has exactly two axes of symmetry? What about three axes of symmetry? What about n axes of symmetry for $n \geq 4$?

Problem 2. Can a bounded plane figure have two axes of symmetry? What about unbounded? Can it have exactly two parallel axes of symmetry?

Problem 3. How many axes of symmetry can a triangle have?

More Difficult Problems

Problem 4. There are 81 grid points (uniformly spaced) in the square shown in the diagram below, including the points on the edges. Point P is in the center of the square. Given that point Q is randomly chosen among the other 80 points, what is the probability that the line PQ is a line of symmetry for the square?



Problem 5. The point $P(a, b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1, 5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6, 3)$. What is $b - a$?

Problem 6. A triangle with vertices $A(0,2)$, $B(-3,2)$, and $C(-3,0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by $90°$ to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$?

(A) counterclockwise rotation about the origin by $90°$.
(B) clockwise rotation about the origin by $90°$.
(C) reflection about the x-axis
(D) reflection about the line $y = x$
(E) reflection about the y-axis.

Problem 7. Triangle OAB has $O = (0,0)$, $B = (5,0)$, and A in the first quadrant. In addition, $\angle ABO = 90°$ and $\angle AOB = 30°$. Suppose that OA is rotated $90°$ counterclockwise about O. What are the coordinates of the image of A?

Problem 8. Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

Problem 9. Let T be the triangle in the coordinate plane with vertices $(0,0), (4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of $90°, 180°$, and $270°$ counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a $180°$ rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a $90°$ rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

Problem 10. Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12$. A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. Let ℓ be the length of the light path from the time it leaves point A until it next reaches a vertex of the cube. Find ℓ.