Geometric Inequalities!

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1 Good to Know

- **The Triangle Inequality.** Let ABC be a triangle. Then $AB + AC \geq BC$ (and the two sister statements). Equality holds if and only if A, B and C are collinear.
- **Ptolemy's Inequality.** Let A, B, C and D be three points in the Euclidean space. Then

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Equality holds if and only if A, B, C, D are coplanar and concyclic in this order.

Walkthrough.

(a) Keeping the equality case in mind, you should seek a way to reduce the problem from the three-dimensional case to its planar cousin. How could you possibly do this?

(b) If you know inversion, you're hitting a home run. Try to invert around one of the vertices of the quadrilateral.

(c) Otherwise (without knowledge of inversion) the proof might seem a little unmotivated; but nevertheless, it's very cool. Let E be a point so that $\triangle ADC \simeq \triangle AEB$. What other pair of similar triangles can you find? What's going on is a phenomenon called *spiral similarity.*

(d) Find a way to relate the length products involved in the inequality's statement to the ratios derived from the above similarities.

(e) Finish by the triangle inequality.

Fact. The two results are actually equivalent and equally as powerful.

2 Problems

The problems in this set are supposed to have a geometric flavor, as opposed to an algebraic one. That being said, it's a good idea to try to focus on the "geometry" of each problem, instead of trying to immediately come up with estimates. Remember that geometric inequalities are very silly problems. I mean it. My favorites are $4, 9, 11$ and 14 (with 14 being by favorite problem of all time).

2.1 Appetizers

1. Let ABC be a triangle and let M be the midpoint of BC. Prove that $AM < \frac{AB+AC}{2}$.

2. a. Let ABC be a triangle with an obtuse angle at A. Let M be the midpoint of side BC. Prove that $AM < \frac{1}{2}BC$.

   b. (Romania JTST 2021). Let $ABCD$ be a convex quadrilateral with obtuse angles at A and C. On sides AB, BC, CD and DA, consider the points K, L, M and N respectively. Prove that the perimeter of $KLMN$ is greater than or equal to $2AC$.

3. Point P lies inside of triangle ABC. Prove that

$$AP + BP + CP < AB + BC + CA.$$

4. (IGO 2021). Let $ABCD$ be a parallelogram. Points E, F lie on the sides AB, CD respectively, such that $\angle EDC = \angle FBC$ and $\angle ECD = \angle FAD$. Prove that $AB \geq 2BC$.

5. (MDC 2016). Let ABC be a triangle and let D the foot of the altitude from A and M the midpoint side BC. Let S be a point inside segment DM and let P and Q the projections of S onto the lines AB and AC respectively. Prove that the length of the segment PQ does not exceed one quarter the perimeter of triangle ABC.

2.2 Main Course

6. (RMM 2023). Given an acute triangle ABC, let H and O be the orthocenter and circumcenter, respectively. Let K be the midpoint of the line segment AH. Also let ℓ be a line through O, and let P and Q be the orthogonal projections of B and C onto ℓ, respectively. Prove that $KP + KQ \geq BC$.

7. (Wait, what?) Let a, b, c, d be positive real numbers. Prove that

$$\sqrt{(a^2 - ab + b^2)(c^2 - cd + d^2)} + \sqrt{(a^2 + ad + d^2)(c^2 + cb + b^2)} \geq (a + c)(b + d).$$

8. (Putnam 1972) Let A, B, C, D be non-coplanar such that $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$. Prove that $AB = CD$ and $AD = BC$.

9. (Romania EGMO TST 2023). Let D be a point inside the triangle ABC. Let E and F be the projections of D onto AB and AC, respectively. The lines BD and CD intersect the circumcircle of ABC the second time at M and N, respectively. Prove that

$$\frac{EF}{MN} \geq \frac{r}{R},$$

where r and R are the inradius and circumradius of ABC, respectively.

10. (ISL 2011). Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

11. (ISL 2014). Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

2.3 Dessert

12. (IZHO 2022). In triangle ABC, a point M is the midpoint of AB, and a point I is the incentre. Point A_1 is the reflection of A in BI, and B_1 is the reflection of B in AI. Let N be the midpoint of A_1B_1. Prove that $IN > IM$.

13. (Romania EGMO TST 2022). Let $ABCD$ be a convex quadrilateral and let O be the intersection of its diagonals. Let P, Q, R, and S be the projections of O on AB, BC, CD, and DA respectively. Prove that

$$2(OP + OQ + OR + OS) \leq AB + BC + CD + DA.$$

14. (IMO 2002). Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$