UT Arlington Mid-Cities Math Circle $(MC)^2$
Number Theory and Digits
January 25, 2023

Warm-up problems

Problem 1. Tom writes all the numbers from 1 to 20 in a row and obtains the 31-digit number 1234567891011121314151617181920. Then he deletes 24 of the 31 digits such that the remaining number is as large as possible. Which number does he get?

Problem 2. Each of eight, consecutive, three-digit numbers is divisible by its last digit. What is the sum of digits of the smallest number?

Problem 3. How many ten-digit numbers are there which contain only the digits 1, 2 or 3, and in which every pair of adjacent digits differs by 1?

More difficult problems

Problem 4. Let $N = 9 + 99 + 999 + \cdots + 999...9$, where the last term in that sum has 321 nines. Determine the sum of the digits of N.

Problem 5. Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

Problem 6. Find the smallest positive integer ending in 1986 which is divisible by 1987.

Problem 7. The integers $1, ..., 2022$ are written in any order and concatenated. Show that the result is always an integer which is not the cube of another integer.

Problem 8. Prove that the digit which is the prior to the last digit in the decimal expansion of 3^n is even.

Problem 9. Find the number of ordered pairs of positive integers (a, b) such that $a + b = 1000$ and neither a nor b has a zero digit.

Problem 10. If n is a positive integer such that the first digit in the decimal expansion of both 2^n and 5^n is x, find x.

Number Theory and Digits II
February 8, 2023

Warm-up problems

Problem 1. Two 3-digit numbers have all their 6 digit distinct. The first digit of the second number is twice the last digit of the first number. What is the smallest possible sum of the two such numbers?

Problem 2. Consider the set of all the 7-digit numbers that can be obtained using, for each number, all the digits $1, 2, 3, ..., 7$. List the numbers of the set in increasing order and split the list exactly at the middle into two parts of the same size. What is the last number of the first half?

More difficult problems

Problem 3. Sarah intended to multiply a two digit number and a three digit number, but she left out the multiplication sign and simply placed the two digit number to the left of the three digit number, thereby forming a five digit number. This number is exactly nine times the product Sarah should have obtained, find the sum of the two digit number and the three digit number.

Problem 4. The number 1.5 is special because it is equal to one quarter of the sum of its digits, as $1 + 5 = 6$ and $\frac{6}{4} = 1.5$. Find all the numbers that are equal to one quarter of the sum of their own digits.

Problem 5. The numbers 1447, 1005 and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

Problem 6. Find the smallest positive integer whose cube ends in 888.

Problem 7. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.

Problem 8. Find the smallest positive integer ending in 1986 which is divisible by 1987.

Problem 9. Find the first digit before and after the decimal point of $\left(\sqrt{2}+\sqrt{3}\right)^{1980}$.

Problem 10. Does there exist a nonconstant polynomial $P(x)$ with integer coefficients, such that, for every positive integer n, the sum of the digits of $|P(n)|$ is not a Fibonacci number? (Recall that the Fibonacci numbers are the elements of the sequence $0, 1, 1, 2, 3, 5, ...$, where each term after the first two terms is the sum of the two terms before it.)

Problem 11. Prove that there are infinitely many composite numbers n, such that n divides $3^{n-1} - 2^{n-1}$.

Problem 12. Prove that there exist infinitely many positive integers n such that the decimal expansion of 2^n ends with n.