UT Arlington Mid-Cities Math Circle $(MC)^2$
Counting
October 5, November 16, 2022

Warm-Up Problems

1. How many 5-letter "words" can one make out of 26 letters, under the condition that a word has at least one vowel?

2. Six horses participate in a race. How many ways are there to finish the race if we assume that no two horses finish simultaneously?

3. A script-writer decided that in his plot, James Bond must have 10 girlfriends, Alice, Barbara, Clara, Denise, Eleanor, Fiona, Georgina, Hannah, Irene, and Janelle; exactly 3 of which are spies. How many different combinations of spies can there be?

More Difficult Problems

4. Each of the faces of a cube is colored by a different color. How many of the colorings are distinct?

5. There are 2022 people must sit at a circular table. How many of the 2022! arrangements are distinct, i.e., do not have the same neighboring relations (reflections and rotations give the same arrangements).

6. Can you arrange the numbers $1, 2, ..., 9$ along a circle, so that the sum of two neighbors are never divisible by 3, 5, or 7?

7. How many ways are there to divide 12 identical oranges between 4 people?

8. The same question as in the previous problem, but everybody gets at least one orange.

9. 2022 points are chosen on a circle. In how many ways can we join pairs of points so that the resulting 1011 chords are pairwise nonintersecting?

10. In how many ways can you place parentheses in a nonassociative product of n factors? For example, if $n = 4$ we have the following 5 different products:
$(((x_1x_2)x_3)x_4), (x_1(x_2(x_3x_4))), ((x_1x_2)(x_3x_4)), ((x_1(x_2x_3))x_4), (x_1((x_2x_3)x_4)).$

11. In a lottery, 6 numbers are selected from the set $\{1, 2, ..., 49\}$. There are $\binom{49}{6}$ possible selections. How many of these selections have at least one pair of consecutive numbers (e.g. $\{1, 4, 6, 7, 20, 48\}$)?

More Warm-Up Problems

12. How many different arrangements can be made using all of the letters in the word "REARRANGE"?

13. Find the number of ways to place 3 rooks on a 5×5 chess board so that no two of them attack each other.

14. A number of men and women seat at a round table. It is known that there are 7 women who have a woman to their right and 12 women who have a man to their right. We know that 3 out of each 4 men have a woman to their right. How many people are seated at the table?

Other Difficult Problems

15. A square 8×8 board is divided into 64 small squares with side-length of 1 cm each. Each small square can be painted black or white. Find the total number of ways to color the board so that every 2×2 square (i.e., a square of side-length 2 cm formed by 4 small squares with a common vertex) has two black squares and two white squares.

16. A spider has 8 feet, 8 different shoes and 8 different socks. Find the number of ways in which the spider can put on the 8 socks and the 8 shoes, **considering the order in which it puts them on**. The only rule is that to put a shoe on the spider must already have a sock on that foot.

    Note: If we disregard the order the spider put the shoes and the socks on, the answer is $(8!)^2$.

17. How many subsets of $\{1, 2, ..., 2022\}$ have no two successive numbers?