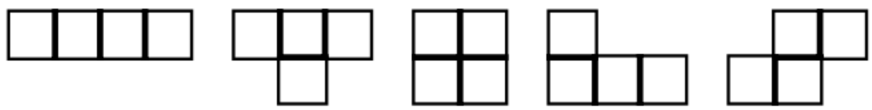
UT Arlington Mid-Cities Math Circle $(MC)^2$
Tiling and Coloring
September 7, 21, 2022

A *tetromino* is a tile composed of four squares, connected along the edges. The five different (up to rotation and reflection) tetrominoes are pictured below and are called from left to right: straight tetromino, T-tetromino, square tetromino, L-tetromino, and skew tetromino.



Warm-up Problems

Problem 1. Can we cover an 8×8 board with L-tetrominoes? How about a 10×10 board?

Problem 2. Two opposite corners of a 4×4 board are removed. Can the remaining figure be covered with dominoes of the shape 1×2? How about an 8×8 board?

Problem 3. All four corners of a 6×6 board are removed. Can the remaining figure be covered with L-tetrominoes? How about an 8×8 board?

More Difficult Problems

Problem 4. Can we cover a 10×10 board with straight tetrominoes?

Problem 5. Can we cover a 10×10 board with T-tetrominoes?

Problem 6. A 7×7 board is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?

Problem 7. Is there a way to pack 250 bricks of dimension $1 \times 1 \times 4$ into a $10 \times 10 \times 10$ box?

Problem 8. Is there a way to pack 53 bricks of dimension $1 \times 1 \times 4$ into a $6 \times 6 \times 6$ box? The faces of the bricks are parallel to the faces of the box.

Problem 9. A 23×23 square is completely tiled by 1×1, 2×2, and 3×3 tiles. What minimum number of 1×1 tiles are needed?

Problem 10. A rectangular room has a floor tiled with tiles of two shapes: 2×2 and 1×4. The tiles completely cover the floor of the room, and no tile has been damaged, or cut in half. One day, a heavy object is dropped on the floor and one of the tiles is cracked. The handyman removes the damaged tile and goes to the storage to get a replacement. But he finds that there is only one spare tile, and it is of the other shape. Can he rearrange the remaining tiles in the room in such a way that the spare tile can be used to fill the hole?

Problem 11. A beetle sits on each square of a 9×9 chessboard. At a signal each beetle crawls diagonally onto a neighboring square. Then it may happen that several beetles will sit on some squares and none on others. Find the minimal possible number of free squares.

Problem 12. (*The Art Gallery Problem*) An art gallery has the shape of a simple n-gon. Find the minimum number of watchmen needed to survey the building, no matter how complicated its shape.

Problem 13. Let $\mathcal{P}$ be a finite set of squares on an infinite chessboard. Kelvin the Frog notes that $\mathcal{P}$ may be tiled with only 1×2 dominoes, while Alex the Kat notes that $\mathcal{P}$ may be tiled with only 2×1 dominoes. The dominoes cannot be rotated in each tiling. Prove that the area of $\mathcal{P}$ is a multiple of 4.

Problem 14. A configuration of 4043 points in the plane is called Texan if it consists of 2021 red points and 2022 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Texan configuration if the following two conditions are satisfied:

(a) No line passes through any point of the configuration.

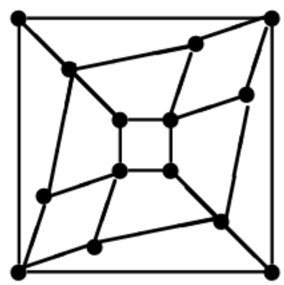
(b) No region contains points of both colors.

Find the least value of k such that for any Texan configuration of 4043 points, there is a good arrangement of k lines.

More Warm-up Problems

Problem 15. Given a 2×5 board and domino tiles of size 2×1, count the number of ways to tile the given board using the domino tiles. What about a 2×10 board?

Problem 16. On the figure below, a road map connecting 14 cities is shown. Is there a path passing through each city exactly once?



Other Difficult Problems

Problem 17. In a 4×4 board the numbers from 1 to 15 are arranged in the following way:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

In a move we can move some number that is in a square sharing a side with the empty square to that square. Is it possible to reach the following position using these moves?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	