Some Selected Problems, at the UTA $(MC)^2$

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Problem 1. Determine the sum of all prime numbers $p < 100$ such that there exists a prime number q such that $100q + p$ is a perfect square.

Problem 2. Determine whether the number

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right)$$

is greater than, less than, or equal to 50.

Problem 3. Convex pentagon $ABCDE$ is inscribed in circle γ. Suppose that $AB = 14$, $BE = 10$, $BC = CD = DE$, and $[ABCDE] = 3[ACD]$. Then there are two possible values for the radius of γ. The sum of these two values is $\sqrt{n}$ for some positive integer n. Determine n.

Problem 4. Let n be an odd positive integer. Some squares (could be all, or none) of an $n \times n$ grid are shaded green. For any four (distinct) cells all sharing a vertex, at most two of them are green. Determine the maximum possible number of green cells.

Problem 5. Determine whether there exist non-integer real numbers a and b with $a > b > 1$ such that, for all positive integers n, we have that $\lceil a^n \rceil \cdot \lceil b^n \rceil$ is a perfect square.

Problem 6. Let $n \geq 3$ be an integer. Prove that there exist odd positive integers x and y such that $2^n = x^2 + 7y^2$.

Problem 7. At a math camp, there are 2022^{2022} students. Each student is friends with three others. The director of the camp wants to arrange all the students in a line so that any two friends have at most 2022 students between them. Can the director necessarily accomplish this?

Problem 8. Let ABC be an acute triangle, and let M be the midpoint of arc BAC of its circumcircle. Point X lies inside the triangle such that $\angle BAX = 2\angle ABX$ and $\angle CAX = 2\angle ACX$. Prove that $XA = XM$.

Problem 9. Let n be a positive integer, and let p be a prime number such that $n < p < \frac{4n}{3}$. Prove that p divides $\sum_{i=0}^{n} \binom{n}{i}^4$.

Problem 10. Let $n \geq 3$ be an integer. Each cell of an $n \times n$ square grid is colored either black or white. In each black cell, we write the number of white cells that share at least one vertex with it. Determine the largest possible value of the sum of the numbers in all the black cells.

Problem 11. Let n be a positive integer, and let $x_1, \ldots, x_n$ be real numbers. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|} \geq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|}.$$