Polynomials, at the UTA $(MC)^2$

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Problem 1. Compute the positive real number x such that $(x^2-1)^2-1=9800$.

Problem 2. Let r and s be the solutions to the equation $2x^2+5x-12=0$. Compute $(r-1)(s-1)$.

Problem 3. Suppose that r, s, and t are pairwise distinct real numbers such that the roots of the polynomial $P(x)=x^3+x^2-r^2x-2022$ are r, s, and t. Compute $P(1)$.

Problem 4. Let $P(x)$ equal

$$\frac{64(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}+\frac{27(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}+\frac{8(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}+\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}.$$

Compute $P(100)$.

Problem 5. Suppose that $x_1<x_2<x_3$ are three real roots of the equation $\sqrt{2014}x^3-4029x^2+2=0$. Compute $x_2(x_1+x_3)$.

Problem 6. Find all polynomials $P(x)$ with real coefficients such that $P(0)=0$ and $P(t^2+1)=P(t)^2+1$ for all real numbers t.

Problem 7. The polynomials $P_1(x)=4x^5-311x^4-704x^3-1255x^2-1964x-2880$ and $P_2(x)=4x^5-279x^4-632x^3-1127x^2-1764x-2592$ have four roots in common. Suppose that s and t are real numbers such that $P_1(s)=P_2(t)=0$ and $P_2(s), P_1(t)\neq 0$. Compute $s+t$.

Problem 8. Determine whether there exists a polynomial $P(x,y)$ with real coefficients such that, for all real numbers s and t, we have that $P(s,t)$ is positive if and only if both s and t are positive.

Problem 9. Let $n\geq 2$ be an integer. Let $P(x_1,x_2,\ldots,x_n)$ be a nonconstant n-variable polynomial with real coefficients. Suppose that whenever $r_1,r_2,\ldots,r_n$ are real numbers, at least two of which are equal, we have $P(r_1,r_2,\ldots,r_n)=0$. Prove that $P(x_1,x_2,\ldots,x_n)$ cannot be written as the sum of fewer than $n!$ monomials. (A monomial is a polynomial of the form $cx_1^{d_1}x_2^{d_2}\ldots x_n^{d_n}$, where c is a nonzero real number and d_1, d_2, …, d_n are nonnegative integers.)

Problem 10. Find all polynomials $P(x)$ with integer coefficients such that for all real numbers s and t, if $P(s)$ and $P(t)$ are both integers, then $P(st)$ is an integer.

Problem 11. Determine whether there exist nonconstant polynomials $P(x)$ and $Q(x)$ with real coefficients such that the equation

$$P(x)^{10}+P(x)^9=Q(x)^{21}+Q(x)^{20}$$

holds.