Graph Theory, at the UTA Mid-Cities Math Circle

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Problems:

Problem 1. A convex polyhedron has 50 faces, all of which are triangles. How many vertices does it have?

Problem 2. In a round-robin basketball tournament, each basketball team plays every other basketball team exactly once. If there are 20 basketball teams, what is the greatest number of basketball teams that could have at least 16 wins after the tournament is completed?

Problem 3. In the city of Königsberg there is a river that surrounds islands A and B and splits the two sides C and D of the mainland. There are two bridges connecting C and A, two bridges connecting D and A, one bridge connecting C and B, one bridge connecting D and B, and one bridge connecting A and B. (See the diagram on the board.) Is it possible to start somewhere in the city and tour the city, crossing each bridge exactly once? It is not required that the start and end points are the same. (It is not allowed to use boats, helicopters, etc.)

Problem 4. Prove that every connected graph has a spanning tree. (If you don't know what these words mean yet, don't worry.)

Problem 5. Let $n \geq 3$ be a positive integer. Suppose that n villages are on a plain. Suppose that some pairs of these villages are connected by pathways so that no two pathways intersect except possibly sharing an endpoint. What is the maximum possible number of pathways?

Problem 6. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wishes to build two-way roads that connect certain pairs of cities such that for each city C and each integer $1 \leq i \leq 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

a) Prove that if n is odd then Mark cannot achieve his wish.

b) Prove that if n is even then Mark can achieve his wish.

Problem 7. There are 15 cities, and there is a train line between each pair operated by either the Red Rail Road or the Lemon Locomotive League. A tourist wants to visit exactly three cities by traveling in a loop, all by traveling on one line. What is the minimum number of such 3-city loops?

Problem 8. Let n be a positive integer. Let $d_1, d_2, \ldots d_n$ be positive integers. Suppose that there is a tree with n vertices $v_1, v_2, \ldots v_n$ such that vertex v_i has degree d_i for all $i = 1, 2, \ldots n$.

a) What is $d_1 + d_2 + \ldots + d_n$ in terms of n?

b) Prove that there exists i such that $d_i = 1$.

c) Using part b, find a recursive formula for the number of such trees.

d) Find a better form for the formula from part c, and use it to show that the total number of trees with vertices $v_1, v_2, \ldots v_n$ is n^{n-2}.

Harder Problems:

Problem 9. Let n and r be positive integers. Consider all graphs with n vertices such that there do not exist $r + 1$ vertices with the property that every two of them are connected by an edge. Among these graphs, which one has the most edges?

Problem 10. In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it. We wish to close some of Onewaynia's roads in such a way that every city now has exactly one road leaving it and exactly one road entering it. Prove that the number of ways of doing this is of the form 2^n for some positive integer n.

Problem 11. The Frontier Lands have 50 towns, some pairs of which are directly connected by Morton's railroad tracks (which are bidirectional and may pass over each other), and it is possible to travel from any town to any other town via these tracks, possibly stopping at other towns on the way. Morton decides that he wants some tracks destroyed so that each town is directly connected to an odd number of other towns. (After Morton destroys the tracks, it might no longer be possible to travel from any town to any other town.) Prove that this is possible.

Problem 12. Let T be a tree with n vertices, for an integer $n > 1$. Suppose that T has exactly k leaves, for some integer k. (A *leaf* in a tree is a vertex of degree 1.) Suppose that there exists a set S of vertices of T such that S contains at least $\frac{n+k-1}{2}$ vertices, and no two vertices in S are adjacent. Prove that the length of the longest path in T is even.

Problem 13. Find the maximum number E such that the following holds: there is a graph with 60 vertices and E edges with each edge colored either red or blue such that there are no monochromatic 3-vertex cycles and no monochromatic 5-vertex cycles. (A cycle is *monochromatic* if all its edges are the same color.)

Problem 14. Let ε be a positive real number. Prove that for all sufficiently large positive integers n, any graph with n vertices and at least $(1 + \varepsilon)n$ edges has two distinct cycles that contain the same number of vertices.