PIGEONHOLE PRINCIPLE

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Jon Erickson

jferickson@ucdavis.edu

The pigeonhole principle, in its simplest form, says that if n items are put into m containers, and $n > m$, then some container must have more than 1 item. For example, if you try to put 5 pigeons into 4 pigeonholes, some pigeonhole must contain at least 2 pigeons. This principle may seems so straightforward as to be trivial, but when applied skillfully, it can be used to solve difficult problems!

One important point is that we are not guaranteed that a container has exactly two items. We are only guaranteed that a container has two **or more** items. In our example with 5 pigeons in 4 pigeonholes, it could be the case that all 5 pigeons are in one pigeonhole!

Additionally, one can state more general versions of the pigeonhole principle. For example, if one places n items into m containers, and $n > mk$ (where k is also a non-negative integer), then some container must have more than k items. If there were k or fewer items in each of the m containers, then the total number of items would be at most mk.

Warmup Problems

(1) Among 13 persons, there are 2 born in the same month.

(2) What is the least number of people in a room needed to guarantee that 2 people have the same birthday? What if we want to guarantee that 3 people have the same birthday? What about 5 people with the same birthday?

(3) If each point of the plane is colored red or blue, then there are two points of the same color at distance 1 from each other.

Problems

(4) You are given some arbitrary points in a 2×2 square. What is the minimum number of points needed to guarantee that there is some pair of points which are at most $\sqrt(2)$ apart?

(5) You are given some arbitrary points in an equilateral triangle of side length 1.

- What is the number m_2 of points needed to guarantee that there is a pair which are at most $\frac{1}{2}$ apart?
- What is the number m_3 of points needed to guarantee that there is a pair which are at most $\frac{1}{3}$ apart?
- For any positive integer n, what is the number of points m_n needed to guarantee that there is a pair of points which are at most $\frac{1}{n}$ apart?

(6) You invite $n > 1$ friends to a party. Some of them shake hands with each other. Prove that there is a pair of people who have shaken hands with the same number of people.

(7) Prove that any $(n+1)$-element subset of $\{1, 2, \ldots 2n\}$ contains two relatively prime integers.

(8) You invite 6 friends to a party. Some of your friends may know each other, and some of your friends may have never met.

- Prove that there is either a group of 3 friends who all know each other, or a group of 3 friends who have never met (Putnam 1953)
- If you invite 5 friends to a party, is this still necessarily the case? Prove or give a counterexample.

(9) The positive integers 1 to 101 are written in any order. Prove that there will always be a increasing subsequence of length 11, or a decreasing subsequence of length 11.

(10) The 7 numbers a, b, c, d, e, f, g are non-negative real numbers, and add to 1. Let M be the maximum of the quantities $a+b+c, b+c+d, c+d+e, d+e+f, e+f+g$. Determine the minimum possible value that m can take. (IMO Shortlist 1981)

(11) There are 17 scientists who correspond by email. They all talk to one another, and each pair only discusses one of three topics: algebra, combinatorics, and topology. Prove that there is a group of 3 people who all correspond about the same topic (IMO 1964)

(12) Five lattice points are chosen in the plane (i.e., five points with integer coordinates). Prove that you can always choose two points such that the segment joining them passes through a third lattice point.

(13) A chess player has 77 days to prepare for a tournament. He wants to play at least 1 game per day, but not more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

(14) Prove that for $n \geq (p-1)(q-1)+1$, any sequence of n distinct integers will have an increasing subsequence of length p, or a decreasing subsequence of length q.

(15) Each point of an equilateral triangle is colored red or blue. Prove that there exist three points of the same color forming a right triangle, or give a counterexample.

(16) Prove that given seven real numbers $y_1, y_2, \ldots, y_7$, there are at least two of them (y_i and y_j) such that

$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \frac{1}{\sqrt{3}}$$