Inequalities, at the UTA $(MC)^2$

Luke Robitaille

December 15, 2021

SOME BASIC INEQUALITIES:

Trivial Inequality: For all real numbers x,
$$x^2 \geq 0.$$

AM-GM Inequality: For nonnegative real numbers $a_1, \ldots, a_n$,
$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}.$$

Cauchy-Schwarz Inequality: For real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$,
$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

PROBLEMS:

Problem 1. Let x and y be positive real numbers such that $x + y = 1$. Find the largest possible value of xy.

Problem 2. Let x and y be positive real numbers such that $x + \frac{1}{y} = 7$. Find the smallest possible value of $y + \frac{1}{x}$.

Problem 3. Let x and y be positive real numbers such that $x + y = 1$. Find the largest possible value of $x^2 y$.

Problem 4. Let x and y be positive real numbers such that $x + y = 1$. Find the smallest possible value of $x^8 + y^8$.

Problem 5. Find the least possible value of $(xy - 1)^2 + (x + y)^2$ for real numbers x and y.

Problem 6. Let $w, x, y,$ and z be positive real numbers. Prove that $w^{13} + x^{13} + y^{13} + z^{13} \geq (wxyz)^3(w + x + y + z)$.

Problem 7. Let $x, y,$ and z be real numbers. Prove that $x^2 + y^2 + z^2 \geq xy + xz + yz$.

Problem 8. Let $x, y,$ and z be positive real numbers. Prove that
$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}.$$

Problem 9. Let $a, b, c,$ and d be positive real numbers such that $(a + c)(b + d) = ac + bd$. Find the smallest possible value of
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem 10. Let $x_1, \dots, x_{2021}$ be positive real numbers. Prove that

$$\frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_1 x_2}{x_3} + \dots + \frac{x_1 x_2 \dots x_{2020}}{x_{2021}} \geq 4(1 - x_1 x_2 \dots x_{2021}).$$

Problem 11. Let n be a positive integer. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be real numbers such that

$$(a_1^2 + \dots + a_n^2 - 1)(b_1^2 + \dots + b_n^2 - 1) > (a_1 b_1 + \dots + a_n b_n - 1)^2.$$

Prove that $a_1^2 + \dots + a_n^2 > 1$ and $b_1^2 + \dots + b_n^2 > 1$.

Problem 12. Let $a_1, a_2, a_3, \dots$ be an infinite sequence of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

Find the largest possible value of the sum

$$\sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \dots a_n)^{\frac{1}{n}}.$$

Problem 13. Let $n > 1$ be an integer. Let $x_1 \geq x_2 \geq \dots \geq x_n$ and $y_1 \geq y_2 \geq \dots \geq y_n$ be real numbers such that $x_1 + \dots + x_n = y_1 + \dots + y_n = 0$ and $x_1^2 + \dots + x_n^2 = y_1^2 + \dots y_n^2 = 1$. Prove that

$$(x_1 y_1 + x_2 y_2 + \dots + x_n y_n) - (x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1) \geq \frac{2}{\sqrt{n-1}}.$$