Problems Involving Digits, at the UTA $(MC)^2$

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Problem 1. Let $N = 9 + 99 + 999 + \cdots + 999\ldots9$, where the last term in that sum has 321 nines. Determine the sum of the digits of N.

Problem 2.
(a) Does there exist a positive integer n such that the sum of the digits of n^2 is equal to 2021?
(b) What if we change 2021 to 2020?

Problem 3. In the equation $PIANO+SONATA = FIFTEEN$, each letter represents a digit. Each letter represents the same digit everywhere it appears, and no two letters represent the same digit. (For example, the three N's all represent the same digit, which is a different digit than what the P represents.) Also, numbers are not allowed to start with the digit zero. Determine what digit each letter in the equation represents.

Problem 4. Let a be a nonnegative integer. Let b be the product of the digits of a, and let c be the product of the digits of b. Suppose that $b \geq 10$, and c is odd. Determine all possible values of the units digit of b.

Problem 5. Does there exist a positive integer n such that all the digits of n are greater than 5, and all the digits of n^2 are less than 5?

Problem 6. Let p be a prime number. Alice and Bob are playing a game. There is a row of p boxes written on the chalkboard. Initially all the boxes are empty. The players alternate turns, with Alice going first. On each turn, the player whose turn it is chooses an empty box and writes a digit in that box. Let N be the nonnegative formed by the digits once all the boxes are filled. (It is allowed for N to have leading zeroes.) Alice wins if N is divisible by p, and Bob wins otherwise. Assuming both players play optimally, who wins? (The answer is allowed to depend on p.)

Problem 7. Carmen and Dante play a game. They start with an empty chalkboard. They alternate turns, with Carmen going first. On Carmen's first turn, she chooses a digit and writes it on the board. Thereafter, the player whose turn it is chooses a digit and writes it anywhere in the number on the board, including possibly at the beginning or end. (The number is allowed to have leading zeroes.) If at any point the board is not empty, and the number on the board is a perfect square (i.e., the square of an integer), then the game ends and Dante wins. Does Dante have a strategy to guarantee that he will win in finitely many turns?

Problem 8. Does there exist a nonconstant polynomial $P(x)$ with integer coefficients, such that, for every positive integer n, the sum of the digits of $|P(n)|$ is not a Fibonacci number? (Recall that the Fibonacci numbers are the elements of the sequence $0, 1, 1, 2, 3, 5, \ldots$, where each term after the first two terms is the sum of the two terms before it.)