Some Combinatorics Problems, at the UTA $(MC)^2$

Luke Robitaille

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Problem 1. Determine the number of ways to color each vertex of a cube red or blue so that the color of each vertex is the color of the majority of the three vertices adjacent to it.

Problem 2. Let A be a 90-element subset of the set $\{1, 2, \ldots, 100\}$. Let s be the sum of the elements of A. Determine the number of possible values of s.

Problem 3. Alice has 10 distinguishable twenty-dollar bills and 18 distinguishable fifty-dollar bills. Determine the number of ways for Alice to choose a set of these bills such that the total value of the bills she chooses is exactly 1000 dollars.

Problem 4. There is a 2×2 grid of squares. Each square can be colored with one of 10 colors. Compute the number of distinguishable ways to color the grid. In this problem, two grids look the same if one can be rotated (without flipping it over) to look like the other.

Problem 5. Bob and Carol each have a randomly shuffled deck of eight cards, four red and four black. Every turn, each player places down the two topmost cards of their decks. A player can thus play one of three pairs: two black cards, two red cards, or one of each color. Determine the probability that Bob and Carol play exactly the same pairs as each other for all four turns.

Problem 6. Dave and Erma are playing a game. Dave starts with \$20 and Erma starts with \$21. On each round, if one player has all the money, then that player wins and the game ends; otherwise, a fair coin is flipped. Then, if Dave has x dollars and Erma has y dollars, if the coin comes up heads, Dave pays Erma $\min(x, y)$ dollars; if the coin comes up tails, Erma pays Dave $\min(x, y)$ dollars. They continue playing rounds like this until someone wins. Determine the probability that Dave wins.

Problem 7. Let $n \geq 3$ be an integer. Felipe and Gina are playing a game. Initially, n points are given on a circle. The players alternate turns, with Felipe going first. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. Determine who has a winning strategy. (The answer is allowed to depend on n.)

Problem 8. There are 2021 people sitting around a table. A move consists of two adjacent people switching seats. We want to make some moves so that each person ends up 1000 seats to the left of where they started. Determine the smallest possible number of moves to accomplish this.

Problem 9. Let k and n be integers with $1 \leq k < n$. Hongyi and Isabelle play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Hongyi playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. Determine who has a winning strategy. (The answer is allowed to depend on n and k.)

Problem 10. Let $n > 1$ be an integer. Jamaal has n wooden blocks, numbered 1 through n. He arranges them in a row on a table in arbitrary order. Kate is allowed to make some moves on the row of blocks. Namely, she can choose two blocks in the row with no blocks between them such that the number on the left block is less than the number on the right block, and then she chooses one of the two blocks and removes it from the row. She is allowed to make this move repeatedly. Determine the number of possible orderings at the start such that Kate can make a series of moves to end up with only one block in the row. (For example, if $n = 3$, the arrangement 132 works, because Kate can delete the 3 and then the 2, but the arrangement 321 fails, because Kate is stuck and has no legal move.)

Problem 11. Determine the number of permutations $a_1, a_2, \ldots, a_{10}$ of $1, 2, \ldots, 10$ that satisfy

$$1a_1 \leq 2a_2 \leq \cdots \leq 10a_{10}.$$

Problem 12. At a party, there are 100 cats. Each pair of cats flips a coin, and they shake paws if and only if the coin comes up heads. It is known that exactly 4900 pairs of cats shook paws. After the party, each cat is independently assigned a "happiness index" uniformly at random in the interval $[0, 1]$. We say a cat is practical if it has a happiness index that is strictly greater than the index of every cat with which it shook paws. Determine the expected value of the number of practical cats.

Problem 13. Let S be the set $\{1, 2, 3, \ldots, 20\}$. For any function $f : S \to S$, and for any positive integer k, let $s(f, k) = \sum_{i=1}^{20} f^k(i)$. (Here $f^k(i)$ denotes the result when f is applied k times to i, so $f^1(i) = f(i)$ and $f^{m+1}(i) = f(f^m(i))$.) Compute the smallest positive integer p such that, for any function $f : S \to S$, the sequence $s(f, 1), s(f, 2), s(f, 3), \ldots$ is eventually periodic with period at most p.

Problem 14. There are 2021 small holes in a circular arrangement around an oak tree. Each hole has a walnut in it. The walnuts are numbered $1, 2, \ldots, 2021$ in arbitrary order. Laszlo the squirrel makes 2021 moves as follows: for each $k = 1, 2, \ldots, 2021$, on the kth turn, Laszlo takes the two walnuts adjacent to the walnut numbered k, and swaps their positions. Prove that there exists some number m such that, on Laszlo's mth move, Laszlo swaps some walnuts numbered a and b such that $a < m < b$.

Problem 15. A group of 3366 film critics are voting on the Oscars. Each critic votes for one actor and one actress. It turns out that, for each $k = 1, 2, \ldots, 100$, there exists some actor or actress who received exactly k votes. Prove that there exist two critics who voted for the same actor and the same actress.

Problem 16. Let A be a set of (not necessarily positive) integers, and let $m > 1$ be an integer. Suppose that, for each $i = 1, \ldots, m$, there exists a nonempty subset of A whose elements sum to exactly m^i. Prove that A has at least $\frac{m}{2}$ elements.