Harder problems

Problem 10. Find all positive integers $M$ such that the sequence $a_0, a_1, a_2, \ldots$ defined by

$$a_0 = M + \frac{1}{2} \quad \text{and} \quad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for} \quad k = 0, 1, 2, \ldots$$

contains at least one integer term.

Problem 11. Let $n$ and $k$ be positive integers with $k \geq 2$. Suppose that $a_1, \ldots, a_k$ are pairwise distinct (i.e., all different) elements of the set $\{1, 2, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k - 1$. Show that $n$ does not divide $a_k(a_1 - 1)$.

Problem 12. Determine all positive integers $n$ such that $2011^n + 12^n + 2^n$ is a perfect square.

Problem 13. Denote by $\mathbb{N}$ the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $m^2 + f(n)$ divides $mf(m) + n$ for all positive integers $m$ and $n$.

Problem 14. Determine all positive integers that can be written in the form

$$\frac{\text{lcm}(x, y) + \text{lcm}(y, z)}{\text{lcm}(x, z)}$$

for some positive integers $x$, $y$, and $z$.

Problem 16. Let $p \geq 2$ be a prime number. Alice and Bob play a game. Alice moves first, and then they alternate turns. On each move, the current player chooses an index $i$ in the set $\{0, 1, 2, \ldots, p - 1\}$ that has not been chosen before by either of the two players and then chooses an element $a_i$.
from the set \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). The game ends once all the indices have been chosen. At that point, the following number is computed:

\[
N = a_0 + 10a_1 + 10^2a_2 + \cdots + 10^{p-1}a_{p-1} = \sum_{i=0}^{p-1} a_i \cdot 10^i.
\]

The goal of Alice is to make \( N \) divisible by \( p \), and the goal of Bob is to prevent this. Prove that Alice has a winning strategy.

**New Problems**

**Problem 17.** Given that a four-digit number \( x = \overline{aabb} \) is a perfect square where \( a \) and \( b \) are distinct nonzero digits, find \( x \).

**Problem 18.** Given positive integers \( a, b, c \) such that \( a, b \geq c \) prove that there are positive integers \( x, y \) such that \( x \) has \( a \) digits, \( y \) has \( b \) digits, and \( \gcd(x, y) \) has \( c \) digits.

**Problem 19.** Prove that \( y^2 = x^3 + 7 \) has no integer solutions.

**Problem 20.** A positive integer is called fancy if it can be expressed in the form

\[
2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},
\]

where \( a_1, a_2, \ldots, a_{100} \) are non-negative integers that are not necessarily distinct. Find the smallest positive integer \( n \) such that no multiple of \( n \) is a fancy number.

**Problem 21.** Does there exist a positive integer \( n \) such that all digits of \( n \) are larger than 5 and all digits of \( n^2 \) are smaller than 5?

**Problem 22.** Let \( P(x) \) be a nonconstant polynomial with integer coefficients that has no integer roots. Prove that there is a positive integer \( m \leq 3 \cdot \deg P \) such that \( P(m) \) does not divide \( P(m+1) \).

**Problem 23.** A finite set \( S \) of positive integers has the property that, for each \( s \in S \), and each positive integer divisor \( d \) of \( s \), there exists a unique element \( t \in S \) satisfying \( \gcd(s, t) = d \). (The elements \( s \) and \( t \) could be equal.)

Given this information, find all possible values for the number of elements of \( S \).