UT Arlington Mid-Cities Math Circle $(MC)^2$
Selected Number Theory Problems, Part II
April 28, 2021

Harder problems

Problem 10. Find all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Problem 11. Let n and k be positive integers with $k \geq 2$. Suppose that $a_1, \ldots, a_k$ are pairwise distinct (i.e., all different) elements of the set $\{1, 2, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Show that n does not divide $a_k(a_1 - 1)$.

Problem 12. Determine all positive integers n such that $2011^n + 12^n + 2^n$ is a perfect square.

Problem 13. Denote by $\mathbb{N}$ the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $m^2 + f(n)$ divides $mf(m) + n$ for all positive integers m and n.

Problem 14. Determine all positive integers that can be written in the form

$$\frac{\operatorname{lcm}(x, y) + \operatorname{lcm}(y, z)}{\operatorname{lcm}(x, z)}$$

for some positive integers x, y, and z.

Problem 16. Let $p \geq 2$ be a prime number. Alice and Bob play a game. Alice moves first, and then they alternate turns. On each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that has not been chosen before by either of the two players and then chooses an element a_i

from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The game ends once all the indices have been chosen. At that point, the following number is computed:

$$N = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^{p-1} a_{p-1} = \sum_{i=0}^{p-1} a_i \cdot 10^i.$$

The goal of Alice is to make N divisible by p, and the goal of Bob is to prevent this. Prove that Alice has a winning strategy.

New Problems

Problem 17. Given that a four-digit number $x = \overline{aabb}$ is a perfect square where a and b are distinct nonzero digits, find x.

Problem 18. Given positive integers a, b, c such that $a, b \geq c$ prove that there are positive integers x, y such that x has a digits, y has b digits, and $\gcd(x, y)$ has c digits.

Problem 19. Prove that $y^2 = x^3 + 7$ has no integer solutions.

Problem 20. A positive integer is called fancy if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \cdots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Problem 21. Does there exist a positive integer n such that all digits of n are larger than 5 and all digits of n^2 are smaller than 5?

Problem 22. Let $P(x)$ be a nonconstant polynomial with integer coefficients that has no integer roots. Prove that there is a positive integer $m \leqslant 3 \cdot \deg P$ such that $P(m)$ does not divide $P(m+1)$.

Problem 23. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s, t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.