UT Arlington Mid-Cities Math Circle $(MC)^2$
Selected Number Theory Problems
April 14, 2021

Warm-up problems

Problem 1. Prove that 7 divides $3^{105} + 4^{105}$.

Problem 2. Do there exist positive integers x and y such that $x^3+y^3 = 468^4$?

Problem 3. If n is an even number, prove that 323 divides $20^n+16^n-3^n-1$.

Problem 4. Prove that 641 divides $2^{32} + 1$ (without using a calculator).

More difficult problems

Problem 5. Two players A and B play the following game. Start with $n = 2$. Each player add a proper divisor of n to the current n. The goal is to obtain a number greater or equal to 2021. Who wins?

Problem 6. Find all integers x, y, z that satisfy the equation

$$5x^3 + 11y^3 + 13z^3 = 0$$

Problem 7. Determine the number of ordered pairs (m, n) of positive integers such that $m^2n = 20^{21}$.

Problem 8. Find the largest positive integer n such that $n + 10$ divides $n^3 + 100$ (i.e., such that $n^3 + 100 = m(n + 10)$ for some integer m).

Harder problems

Problem 9. Prove that the equation

$$x^2 + y^2 + z^2 = 3xyz$$

has infinitely many integer solutions.

Problem 10. Find all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Problem 11. Let n and k be positive integers with $k \geq 2$. Suppose that $a_1, \ldots, a_k$ are pairwise distinct (i.e., all different) elements of the set $\{1, 2, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Show that n does not divide $a_k(a_1 - 1)$.

Problem 12. Determine all positive integers n such that $2011^n + 12^n + 2^n$ is a perfect square.

Problem 13. Denote by $\mathbb{N}$ the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $m^2 + f(n)$ divides $mf(m) + n$ for all positive integers m and n.

Problem 14. Determine all positive integers that can be written in the form

$$\frac{\operatorname{lcm}(x, y) + \operatorname{lcm}(y, z)}{\operatorname{lcm}(x, z)}$$

for some positive integers x, y, and z.

Problem 15. Let b and n be positive integers. Suppose that for every positive integer k, there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that there exists a positive integer A such that $b = A^n$.

Problem 16. Let $p \geq 2$ be a prime number. Alice and Bob play a game. Alice moves first, and then they alternate turns. On each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that has not been chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The game ends once all the indices have been chosen. At that point, the following number is computed:

$$N = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^{p-1} a_{p-1} = \sum_{i=0}^{p-1} a_i \cdot 10^i.$$

The goal of Alice is to make N divisible by p, and the goal of Bob is to prevent this. Prove that Alice has a winning strategy.