INDUCTION

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Proof by induction is a technique that allows mathematicians to prove certain types of "infinite" statements: typically, a statement that for each natural number n, a particular property holds. Induction is an essential proof technique, and is used in a wide variety of problems and areas. Generally, an inductive proof proceeds in the following way:

- **Base Case:** Prove that the result holds for the base case. This is often when $n = 0$ or $n = 1$, but could be some other number–for example, if you were proving a claim about polygons, you might want to start with $n = 3$, since the triangle is the polygon with the least number of sides.
- **Induction Hypothesis:** Let n be a natural number which is strictly greater than your base case. The induction hypothesis for n is the claim that the desired result holds for every natural number from your base case to $n - 1$ (for example, if your base case was for $n = 1$, you would assume that the result held for $1, 2, \ldots, n - 1$.)
- **Inductive Step:** Suppose that the induction hypothesis holds for n. Then, prove that the desired result must hold for n.

To see this in action, we will look at an inductive proof of the classic result that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

As a note, it is sometimes the case that in the inductive step, one only needs the desired result to hold for $n-1$ to prove that it holds for n (instead of requiring that it hold for every natural number from the base case to $n - 1$). However, there are times when you will need the full strength of the induction hypothesis to prove a particular result.

Problems

(1) Prove inductively that

$$1^2 + 2^2 + 3^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}$$

(2) Prove inductively that

$$1 + 2 + 4 + 8 + \dots 2^{n-1} = 2^n - 1$$

(3) Prove inductively that

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\dots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

(4) Find and prove (inductively) a formula for the sum of the first n odd numbers.

(5) Prove inductively that for $x > -1, x \neq 0$, and n a natural number greater than 1,

$$(1+x)^n > 1 + nx$$

(6) In Noetheria, the only coins are 3 credit and 5 credit coins. Prove that any amount greater than 7 credits can always be paid in coins.

Induction is often useful for proving properties of structures that are defined **recursively**: such structures are constructed in stages, with each stage (potentially) referring to a previous stage. One famous recursive structure is the Fibonacci numbers. This sequence of numbers is defined as follows:

- F(1) = 1
- F(2) = 1
- For $n > 2$, $F(n) = F(n-1) + F(n-2)$

Hence the sequence of Fibonacci numbers is $1, 1, 2, 3, 5, 8, 13, 21, \dots$.

(7) Find the least number k such that $F(k) \geq 2k$. Prove that for $n \geq k, F(n) \geq 2n$.

(8) Prove that

$$F(n-1)F(n+1) = F(n)^2 + (-1)^n$$

(9) Prove that

$$F(1)^2 + F(2)^2 + \dots F(n)^2 = F(n)F(n+1)$$

(10) Prove that the sum of the interior angles in an n-gon (an n-sided polygon) is $180(n-2)$ degrees.

(11) Prove using induction that the number of diagonals in an n-gon is $\frac{n(n-3)}{2}$. Can you find a direct proof that does not utilize induction?

(12) **Pick's Theorem** Let P be a polygon whose vertices lie on lattice points in the plane (recall that lattice points are points with integer coordinates). Let i be the number of lattice points in the interior of P, and let b be the number of lattice points on P itself (i.e., on the boundary of the polygon). Then the area of the polygon P is

$$A = i + \frac{b}{2} - 1$$

(13) **AM-GM-HM** Let $S = a_1, a_2, \dots a_n$ be a sequence of positive numbers. Recall the definitions of the arithmetic, geometric, and harmonic means:

$$A(S) = \frac{a_1 + a_2 + \dots a_n}{n}$$

$$G(S) = \sqrt[n]{a_1 a_2 \dots a_n}$$

$$H(S) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_n}}$$

Prove that

$$A(S) \geq G(S) \geq H(S)$$

Hint: Prove first that $A(S) \geq G(S)$ holds when $n = 2^k$ using induction on k. Then, prove it for arbitrary n. Finally, use $A(S) \geq G(S)$ to prove $G(S) \geq H(S)$.