Even More Complex Numbers, at the UTA $(MC)^2$

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March 17, 2021

Note: Some of these problems are leftovers from previous weeks' handouts, while others (specifically, problems 18 through 20) are new this week. That's why the numbering system is so strange.

Problems:

Problem 3. Define a sequence $z_0, z_1, z_2, \ldots$ of complex numbers recursively by $z_0 = \frac{1}{100} + i$ and $z_{n+1} = \frac{z_n + i}{z_n - i}$ for all nonnegative integers n. Let a and b be real numbers such that $z_{1000} = a + bi$. Find $a + b$.

Problem 6. Find the number of ordered quadruples (a, b, c, d) of complex numbers such that, for all complex numbers x and y, we have that $(ax + by)^3 + (cx + dy)^3 = x^3 + y^3$.

Problem 8. Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part. Suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

Problem 15. Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

Problem 18. A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Find the number of groups that could be selected.

Problem 19. For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Harder Problems:

Problem 9. Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z) - 1$ have absolute value 1.

Problem 10. Let w, x, y, and z be complex numbers with $|w| = |x| = |y| = |z| = 1$ and $wxyz + 3 = w + x + y + z$. Prove that at least one of w, x, y, and z is equal to 1.

Problem 16. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Problem 17. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incenter. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Problem 20. Let $S = \{1, \ldots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \ldots, a_n) \in S^n \mid a_1 + \cdots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.