More Complex Numbers, at the UTA $(MC)^2$

Luke Robitaille

March 3, 2021

Note: Some of these Problems are leftovers from last week's handout, while others (specifically, problems 11 through 17) are new this week. That's why the numbering system is so strange.

Problems:

Problem 3. Define a sequence $z_0, z_1, z_2, \ldots$ of complex numbers recursively by $z_0 = \frac{1}{100} + i$ and $z_{n+1} = \frac{z_n + i}{z_n - i}$ for all nonnegative integers n. Let a and b be real numbers such that $z_{1000} = a + bi$. Find $a + b$.

Problem 6. Find the number of ordered quadruples (a, b, c, d) of complex numbers such that, for all complex numbers x and y, we have that $(ax + by)^3 + (cx + dy)^3 = x^3 + y^3$.

Problem 8. Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part. Suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \leq \theta < 360$. Find θ.

Problem 11. Let z be a complex number with $|z| = 14$. Let P be the polygon in the complex plane whose vertices are z and every w such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Find the area of P.

Problem 12. For some complex number ω with $|\omega| = 5$, there is some real $\lambda > 1$ such that ω, ω^2, and $\lambda\omega$ form an equilateral triangle in the complex plane. Find the value of λ.

Problem 13. Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg\left(\frac{w-z}{z}\right)$. Find the maximum possible value of $\tan^2\theta$. (Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.)

Problem 14. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

Problem 15. Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

Harder Problems:

Problem 9. Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z) - 1$ have absolute value 1.

Problem 10. Let w, x, y, and z be complex numbers with $|w| = |x| = |y| = |z| = 1$ and $wxyz + 3 = w + x + y + z$. Prove that at least one of w, x, y, and z is equal to 1.

Problem 16. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Problem 17. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incenter. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.