More Complex Numbers, at the UTA (MC)²

Luke Robitaille

March 3, 2021

Note: Some of these Problems are leftovers from last week’s handout, while others (specifically, problems 11 through 17) are new this week. That’s why the numbering system is so strange.

Problems:

Problem 3. Define a sequence \( z_0, z_1, z_2, \ldots \) of complex numbers recursively by \( z_0 = \frac{1}{100} + i \) and \( z_{n+1} = \frac{z_n + i}{z_{n-1}} \) for all nonnegative integers \( n \). Let \( a \) and \( b \) be real numbers such that \( z_{1000} = a + bi \). Find \( a + b \).

Problem 6. Find the number of ordered quadruples \((a, b, c, d)\) of complex numbers such that, for all complex numbers \( x \) and \( y \), we have that \((ax + by)^3 + (cx + dy)^3 = x^3 + y^3\).

Problem 8. Let \( P \) be the product of the nonnegative integers \( n \) such that one such value of \( \theta \) is equal to 1. Let \( \omega, \omega^2, \lambda \omega \) form an equilateral triangle in the complex plane. Find the value of \( \lambda \).

Problem 12. For some complex number \( \omega \) with \( |\omega| = 5 \), there is some real \( \lambda > 1 \) such that \( \omega, \omega^2, \omega^3 \), and \( \lambda \omega \) form an equilateral triangle in the complex plane. Find the value of \( \lambda \).

Problem 13. Let \( w \) and \( z \) be complex numbers such that \( |w| = 1 \) and \( |z| = 10 \). Let \( \theta = \arg \left( \frac{w-z}{z} \right) \). Find the maximum possible value of \( \tan^2 \theta \). (Note that \( \arg(w) \), for \( w \neq 0 \), denotes the measure of the angle that the ray from 0 to \( w \) makes with the positive real axis in the complex plane.)

Problem 14. Given \( f(z) = z^2 - 19z \), there are complex numbers \( z \) with the property that \( z, f(z), \) and \( f(f(z)) \) are the vertices of a right triangle in the complex plane with a right angle at \( f(z) \). There are positive integers \( m \) and \( n \) such that one such value of \( z \) is \( m + \sqrt{n} + 11i \). Find \( m + n \).

Problem 15. Let \( z_1 = 18 + 83i \), \( z_2 = 18 + 39i \), and \( z_3 = 78 + 99i \), where \( i = \sqrt{-1} \). Let \( z \) be the unique complex number with the properties that \( \frac{z-z_1}{z_2-z_1} \cdot \frac{z-z_2}{z_3-z_2} \) is a real number and the imaginary part of \( z \) is the greatest possible. Find the real part of \( z \).

Harder Problems:

Problem 9. Find all nonconstant polynomials \( P(z) \) with complex coefficients for which all complex roots of the polynomials \( P(z) \) and \( P(z) - 1 \) have absolute value 1.

Problem 10. Let \( w, x, y, \) and \( z \) be complex numbers with \( |w| = |x| = |y| = |z| = 1 \) and \( wxyz + 3 = w + x + y + z \). Prove that at least one of \( w, x, y, \) and \( z \) is equal to 1.

Problem 16. Let \( ABC \) be an acute triangle with orthocenter \( H \) and circumcircle \( \Gamma \). Let \( BH \) intersect \( AC \) at \( E \), and let \( CH \) intersect \( AB \) at \( F \). Let \( AH \) intersect \( \Gamma \) again at \( P \neq A \). Let \( PE \) intersect \( \Gamma \) again at \( Q \neq P \). Prove that \( BQ \) bisects segment \( EF \).

Problem 17. Let \( ABC \) be a triangle with \( AB = AC \neq BC \) and let \( I \) be its incenter. The line \( BI \) meets \( AC \) at \( D \), and the line through \( D \) perpendicular to \( AC \) meets \( AI \) at \( E \). Prove that the reflection of \( I \) in \( AC \) lies on the circumcircle of triangle \( BDE \).