Grids, at the UTA $(MC)^2$

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January 20, 2021

Problem 1. Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

Problem 2. Robert colors each square in an empty 3 by 3 grid either red or green. Find the number of colorings such that no row or column contains more than one green square.

Problem 3. Compute the number of ways to write the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 in the cells of a 3 by 3 grid such that

- each cell has exactly one number,
- each number goes in exactly one cell,
- the numbers in each row are increasing from left to right,
- the numbers in each column are increasing from top to bottom, and
- the numbers in the diagonal from the upper-right corner cell to the lower-left corner cell are increasing from upper-right to lower-left.

Problem 4. Michelle is at the bottom-left corner of a 6×6 lattice grid, at $(0, 0)$. The grid also contains a pair of one-time-use teleportation devices at $(2, 2)$ and $(3, 3)$; the first time Michelle moves to one of these points she is instantly teleported to the other point and the devices disappear. If she can only move up or to the right in unit increments, in how many ways can she reach the point $(5, 5)$?

Problem 5. Nine fair coins are flipped independently and placed in the cells of a 3 by 3 square grid. Let p be the probability that no row has all its coins showing heads and no column has all its coins showing tails. What is the value of p?

Problem 6. On a 9×9 square lake composed of unit squares, there is a 2×4 rectangular iceberg also composed of unit squares (it could be in either orientation; that is, it could be 4×2 as well). The sides of the iceberg are parallel to the sides of the lake. Also, the iceberg is invisible. Lily is trying to sink the iceberg by firing missiles through the lake. Each missile fires through a row or column, destroying anything that lies in its row or column. In particular, if Lily hits the iceberg with any missile, she succeeds. Lily has bought n missiles and will fire all n of them at once. Let N be the smallest possible value of n such that Lily can guarantee that she hits the iceberg. Let M be the number of ways for Lily to fire N missiles and guarantee that she hits the iceberg. (Note: Rows and columns are distinguishable, but there is no difference between firing from the left of a column and firing from the right; think of the missile as simply hitting each square in a designated row or column. Rotations are distinguished.) Compute $100M + N$.

Problem 7. You have a $2m$ by $2n$ grid of squares coloured in the same way as a standard checkerboard. Find the total number of ways to place mn counters on white squares so that each square contains at most one counter and no two counters are in diagonally adjacent white squares.

Problem 8. How many ways can Rosa fill a 3 by 3 square grid with nonnegative integers such that no nonzero integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

Problem 9. Michael is trying to drive a bus from his home, $(0, 0)$, to school, located at $(6, 6)$. There are horizontal and vertical roads at every line $x = 0, 1, \ldots, 6$ and $y = 0, 1, \ldots, 6$. The city has placed 6 roadblocks on lattice point intersections (x, y) with $0 \leq x, y \leq 6$. Michael notices that the only path he can take that only goes up and to the right is directly up from $(0, 0)$ to $(0, 6)$, and then right to $(6, 6)$. How many sets of 6 locations could the city have blocked?

Problem 10. If we take a 2×100 (or 100×2) grid of unit squares, and remove alternate squares from along side, the remaining 150 squares form a *comb*. Keisha takes a 200×200 grid of unit squares, and chooses k of these squares and colors them so that James is unable to choose 150 uncolored squares which form a comb. What is the smallest possible value of k?

Problem 11. Let n be an odd positive integer. Find the maximum number of cells of an $n \times n$ board that can be colored green such that every 2×2 square of four cells contains at most two green cells.

Problem 12. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be adjacent if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent). The filling is called a garden if it satisfies the following two conditions:

- The difference between any two adjacent numbers is either 0 or 1.
- If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

Problem 13. The integers from 1 to n^2 are put into an $n \times n$ square array with no repetitions. A path is a series of movements from the bottom left to the top right corner of the square, moving only up or to the right (all paths cover exactly $2n - 1$ squares). The weight of a path is the sum of all numbers covered by the path. For any numbered square we can find a path P with largest weight among all paths and a path Q with the smallest weight among all paths. (If there is a tie, choose arbitrarily.) Among all possible ways to number a square, what is the smallest possible difference between the weights of P and Q?

Problem 14. Let $n \geq 3$ be an integer. Find the smallest integer k with the following property: Given an $n \times n$ board in which each square is either white or black, there exists a division of the board along the gridlines into one or more rectangles such that every rectangle contains at most two black squares, and there are at most k rectangles in the division that contain at most one black square.

Problem 15. Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. The value of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called balanced if there exists some $k \geq 1$ such that each row and each column has a value of k. Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

Problem 16. Consider an infinite white plane divided into square cells. Let k be a positive integer. Determine whether it is possible to paint a positive finite number of cells black so that on each horizontal, vertical, and diagonal line of cells there is either exactly k black cells or none at all.

Problem 17. The integers $1, 2, ..., 64$ are written in the squares of an 8 by 8 chess board, such that for each i with $1 \leq i < 64$, the numbers i and $i + 1$ are in squares that share an edge. Determine the smallest possible sum that can appear along the main diagonal (which goes from the upper-left corner of the grid to the lower-right corner of the grid).

Problem 18. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 19. An $n \times n$ table is written on a square piece of cardboard. Henry draws some diagonals in some of the n^2 cells, then uses a knife to cut along the marked diagonals. Henry finds that the resulting piece of cardboard is still connected. Show that at least $2n - 1$ cells were left uncut.

Problem 20. Find the greatest natural number N such that, for any arrangement of the numbers $1, 2, \ldots, 400$ in a chessboard 20×20, there exist two numbers in the same row or column, which differ by at least N.

Problem 21. Let n be a positive integer. A tiling of a $2n \times 2n$ board is a placing of $2n^2$ dominoes such that each of them covers exactly two squares of the board and they cover all the board. Consider now two separate tilings of a $2n \times 2n$ board: one with red dominoes and the other with blue dominoes. We say two squares are red neighbors if they are covered by the same red domino in the red tiling; similarly define blue neighbors. Suppose we can assign a non-zero integer to each of the squares such that the number on any square equals the difference between the numbers on its red and blue neighbors, i.e., the number on its red neighbor minus the number on its blue neighbor. Show that n is divisible by 3.

Problem 22. On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.