UT Arlington Mid-Cities Math Circle $(MC)^2$
Polynomials
December 2, 2020

Unless otherwise stated, in this handout, all polynomials are assumed to have real coefficients.

Division of polynomials. For any polynomials $f(x)$ and $g(x)$ there exist polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x), \quad \deg r < \deg g \text{ or } r(x) = 0.$$

For example, if $f(x) = x^7 - 1$ and $g(x) = x^3 + x + 1$ then the quotient $q(x)$ is $x^4 - x^2 - x + 1$ and the remainder $r(x)$ is $2x^2 - 1$. If $g(x) = x - a$ we have: $f(a) = 0$ if and only if $f(x) = (x - a)q(x)$ for some polynomial $q(x)$

Warm-up problems

Problem 1. Find the remainder of dividing $f(x) = x^{100} - 2x^{51} + 1$ by $g(x) = x^2 - 1$.

Problem 2. Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

Problem 3. Find $f(100)$, where

$$\begin{aligned} f(x) &= 64\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} + 27\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} \\ &\quad +8\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + 1\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \end{aligned}$$

for all real numbers x.

Problem 4. Let $f(x) = ax^2 + bx + c$. Suppose that $f(x) = x$ has no real roots. Show that the equation $f(f(x)) = x$ has also no real solutions.

More difficult problems

Problem 5. Let a, b be integers. Then the polynomial $(x - a)^2(x - b)^2 + 1$ cannot be presented as a product of two polynomials with integral coefficients and degree bigger than 1.

Problem 6. Suppose a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions

$$P(a) = b, P(b) = c, P(c) = a$$

cannot be satisfied simultaneously.

Problem 7. If $a_1, a_2, ..., a_n$ are distinct integers, prove that the polynomial $P(x) = (x - a_1)...(x - a_n) - 1$ it cannot be presented as a product of two polynomials, each of which of degree bigger than 1.

Problem 8. Factor $(1 + x + ... + x^n)^2 - x^n$ as a product of two non-constant polynomials.

Problem 9. Let $P(x)$ be a polynomial of degree n, so that $P(k) = \frac{k}{k+1}$ for $k = 0, ..., n$. Find $P(n + 1)$.

Problem 10. Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

Problem 11. Find all polynomials $P(x)$ such that $P(0) = 0$ and $P(x^2+1) = P(x)^2 + 1$ for all real numbers x.

Problem 12. Find all polynomials $P(x)$ that satisfy the identity

$$P(x)P(x + 1) = P(x^2 + x + 1).$$

Problem 13. Find all polynomials $P(x)$ that satisfy the identity

$$xP(x - 1) = (x - 2020)P(x).$$

Problem 14. Does there exist a polynomial $P(x, y)$ with real coefficients such that its range is exactly the set of positive real numbers?

Problem 15. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 16. Prove that there exists a unique polynomial $P(x)$ with real coefficients such that $xy - x - y | (x + y)^{1000} - P(x) - P(y)$ for all real x, y.

Problem 17. Find all polynomials $p(x)$ with real coefficients that have the following property: there exists a polynomial $q(x)$ with real coefficients such that

$$p(1)+p(2)+p(3)+\cdots+p(n)=p(n)q(n)$$

for all positive integers n.

New problems

Problem 18. (easy) Suppose that $f(x+3)=3x^2+7x+4$ and $f(x)=ax^2+bx+c$. What is $a+b+c$?

Problem 19. In the expansion of

$$\left(1+x+x^2+\cdots+x^{27}\right)\left(1+x+x^2+\cdots+x^{14}\right)^2,$$

what is the coefficient of x^{28}?

Problem 20. The expression

$$(x+y+z)^{2020}+(x-y-z)^{2020}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

Problem 21. A polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that the polynomial P must be the zero polynomial.