Some Number Theory Problems, at the UTA $(MC)^2$

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1 Warm-Up

Problem 1. How many ordered pairs (m, n) of positive integers satisfy $m^2n = 20^{20}$?

Problem 2. How many ordered pairs (a, b) of positive integers satisfy $\text{lcm}(a, b) = 6^5$?

Problem 3. There exist two distinct positive integers, both of which are divisors of 10^{10}, with sum equal to 157. What are they?

Problem 4. There are three eight-digit positive integers which are equal to the sum of the eighth powers of their digits. Given that two of the numbers are 24678051 and 88593477, compute the third number.

Problem 5. For all positive integers n, let $f(n)$ return the smallest positive integer k for which $\frac{n}{k}$ is not an integer. For example, $f(6) = 4$ because 1, 2, and 3 all divide 6 but 4 does not. Determine the largest possible value of $f(n)$ as n ranges over the set $\{1, 2, \ldots, 3000\}$.

Problem 6. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \le a_i < 3$ for all $0 \le i \le n$. Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

Problem 7. If $p, q,$ and r are primes such that $pqr = 7(p + q + r)$, then find $p + q + r$.

Problem 8. Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

2 Problems

Problem 9. Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. What is the value of A?

Problem 10. There are two prime numbers p so that $5p$ can be expressed in the form $\left\lfloor \dfrac{n^2}{5} \right\rfloor$ for some positive integer n. What is the sum of these two prime numbers?

Problem 11. For odd positive integers $n > 1$, define $f(n)$ to be the smallest odd integer greater than n that is not relatively prime to n. (A positive integer a is *relatively prime* to a positive integer b if $\gcd(a, b) = 1$.) Compute the smallest n such that $f(f(n))$ is a positive integer that is not divisible by 3.

Problem 12. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2 + b + c, b^2 + c + a, c^2 + a + b$ to be perfect squares.

Problem 13. Find all pairs (a, b) of positive integers such that $a^{2020} + b$ is a multiple of ab.

Problem 14. For all positive integers n, let

$$f(n) = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2.$$

Compute $f(2019) - f(2018)$. Here $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n.

Problem 15. Compute the number of ordered pairs (m, n) of positive integers that satisfy the equation $\operatorname{lcm}(m, n) + \gcd(m, n) = m + n + 30$.

Problem 16. Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Problem 17. Let a, b, c be positive integers such that $a^2 - bc$ is a square. Prove that $2a + b + c$ is not prime.

Problem 18. Amy and Bob play a game. At the beginning, Amy writes down a positive integer on the blackboard. Then the players alternate turns, with Bob moving first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n - a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Bob guarantee that he will eventually win?

Problem 19. For positive integers n and k, let $\mho(n, k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mho(n, k)}{3^{n+k-7}}.$$

Problem 20. The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

Problem 21. We call a 5-tuple of integers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

3 Harder Problems

Problem 22. Consider the set

$$A = \left\{1 + \frac{1}{k} : k = 1, 2, 3, 4, \cdots\right\}.$$

a) Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

b) For every integer $x \geq 2$ let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different. Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2$, $y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$).

Problem 23. Let S be a nonempty set of positive integers such that, for any (not necessarily distinct) integers a and b in S, the number $ab+1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Problem 24. A deck of cards is given. The deck contains at least two, but finitely many, cards. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. Prove that the numbers on the cards are all equal.

Problem 25. Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is *expensive* if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \ldots (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}.$$

a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.

b) Prove that for every odd positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

Problem 26. Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.