Lengths and Geometry, at the UTA $(MC)^2$

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1 Theory

Theorem 1: For any triangle XYZ, let $[XYZ]$ denote its area.

a) Let ABC be a triangle and let P be a point on side BC. Prove that $\frac{[ABP]}{[ABC]} = \frac{BP}{BC}$.

b) Let DEF be a triangle and let Q and R be points on sides DE and DF, respectively. Prove that $\frac{[DQR]}{[DEF]} = \frac{DQ \cdot DR}{DE \cdot DF}$.

c) Let $KLMN$ be a convex quadrilateral, and let segments KM and LN intersect at point O. Prove that $\frac{[KLM]}{[KMN]} = \frac{LO}{NO}$.

Theorem 2 (Ceva): Let ABC be a triangle. Let points X, Y, Z lie on sides BC, AC, AB, respectively. Assume that $X \neq B, C; Y \neq A, C; Z \neq A, B$. Then lines AX, BY, CZ are concurrent if and only if $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$.

Theorem 3 (Menelaus): Let ABC be a triangle. A line ℓ not passing through A, B, or C intersects lines BC, AC, AB, respectively, at points X, Y, Z. Then $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$.

Remark: How do we have the same equation in Menelaus and Ceva, even though they aren't the same configuration? Can you generalize Ceva to cases when X, Y, Z might lie outside the segments BC, AC, AB, but still on the lines? Can you figure out a version of Menelaus that we could (correctly) state as an "if and only if"? (Some extra conditions on X, Y, Z...?)

Theorem 4 (Angle Bisector Theorem): Let ABC be a triangle. Let the (internal) bisector of $\angle BAC$ intersect side BC at D. Then $\frac{BD}{CD} = \frac{AB}{AC}$.

Theorem 5 (Ptolemy's Theorem): Let $ABCD$ be a convex quadrilateral inscribed in a circle. Then $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

Theorem 6 (Pythagorean Theorem): Let ABC be a triangle such that $\angle BAC = 90°$. Then $AB^2 + AC^2 = BC^2$.

Theorem 7: Let ABC be a triangle inscribed in a circle of radius R. Then the area of triangle ABC is equal to $\frac{AB \cdot AC \cdot BC}{4R}$.

Theorem 8 (Power of a Point): Let A, B, C, and D be four different points (in a plane). Assume that no three of them all lie on a line.

a) Suppose that line segments $\overline{AB}$ and $\overline{CD}$ intersect at point X. Then $XA \cdot XB = XC \cdot XD$ if and only if A, B, C, and D are concyclic.

b) Suppose that lines AB and CD intersect at X; furthermore, suppose that X does not lie on line segment $\overline{AB}$ or line segment $\overline{CD}$. Then $XA \cdot XB = XC \cdot XD$ if and only if A, B, C, and D are concyclic. (Remark: what about segments tangent to circles...?)

Theorem 9 (Stewart's Theorem): Let ABC be a triangle and let D be a point on segment BC. Let $a = BC, b = AC, c = AB, m = BD, n = CD, d = AD$. Then $amn + ad^2 = b^2m + c^2n$. (When written as $man + dad = bmb + cnc$, this can be humorously remembered as "a man and his dad put a bomb in the sink".)

2 Problems

Problem 0. Prove all the theorems in Section 1. (You don't actually have to do this.)

Problem 1. A circle has a chord of length 10, and the distance from the center of the circle to the chord is 5. What is the area of the circle?

Problem 2 (Apollonius's Formula). Let ABC be a triangle and let M be the midpoint of BC. Prove that $AM^2 = \frac{2AB^2+2AC^2-BC^2}{4}$.

Problem 3. Let $ABCD$ be a rectangle and let X be any point in the plane. Prove that $XA^2 + XC^2 = XB^2 + XD^2$.

Problem 4. A point lies inside a square. The distances from the point to the four vertices of the square, in some order, are $6, 21, 27$, and x, for some positive integer x. What is the value of x?

Problem 5. Let ABC be a triangle.

a) Let M_A, M_B, M_C be the midpoints of segments BC, AC, AB, respectively. Prove that AM_A, BM_B, CM_C are concurrent. The point at which they concur is called the *centroid* of ABC, usually called G.

b) Prove that the perpendicular bisectors of segments AB, AC, BC concur. The point at which they concur is called the *circumcenter* of ABC, usually called O; this point is the center of a circle passing through A, B, and C called the *circumcircle* of triangle ABC.

c) Prove that the altitudes of ABC are concurrent; that is, prove that the line through A perpendicular to BC, the line through B perpendicular to AC, and the line through C perpendicular to AB are concurrent. The point at which they concur is called the *orthocenter* of ABC, usually called H.

d) Prove that the (internal) angle bisectors of $\angle A, \angle B$, and $\angle C$ concur. The point at which they concur is called the *incenter* of ABC, usually called I.

Remark. The centroid, circumcenter, and orthocenter are collinear (this is not obvious); the line they determine (for ABC not equilateral) is called the *Euler line* of ABC.

Problem 6. A circle has center O and radius R. A line intersects the circle at two points A and B. Let X be another point on this line. Prove that $XA \cdot XB = |XO^2 - R^2|$. (This is useful for proving Theorem 8 above. Also, does this clarify the remark about "tangents to circles" in Theorem 8?)

Problem 7. In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

Problem 8. A circle passes through points $(3, 4)$, $(6, 8)$, and $(5, 13)$. Compute the length of a tangent segment from $(0, 0)$ to this circle.

Problem 9. Let ABC be a triangle and let P be a point inside. Let AP and BC meet at A_1, BP and AC meet at B_1, and CP and AB meet at C_1. Point A_2 is the reflection of A_1 over the midpoint of BC. Point B_2 is the reflection of B_1 over the midpoint of AC. Point C_2 is the reflection of C_1 over the midpoint of AB. Prove that AA_2, BB_2, and CC_2 are concurrent.

Problem 10. Let ABC be a triangle and let the (internal) bisector of $\angle A$ intersect BC at D. Let $a = BC, b = AC, c = AB$. Prove that $AD^2 = bc\left(1 - (\frac{a}{b+c})^2\right)$.

Problem 11. Let W, X, Y, Z be points in the plane. Assume that $W \neq X$ and $Y \neq Z$. Then WX is perpendicular to YZ if and only if $WY^2 + XZ^2 = WZ^2 + XY^2$.

Problem 12. Let ABC be a triangle and let A_1, B_1, C_1 be points in the plane. Let α, β, γ be, respectively, the lines through A_1, B_1, C_1 perpendicular to BC, AC, AB. Prove that α, β, γ are concurrent if and only if $(BA_1^2 - CA_1^2) + (CB_1^2 - AB_1^2) + (AC_1^2 - BC_1^2) = 0$.

Problem 13. Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

Problem 14. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and the line ED is perpendicular to the line AD. Segment $\overline{AE}$ intersects the circle at point C between A and E. What is the area of $\triangle ABC$?

Problem 15. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

Problem 16. Let $ABCD$ be a convex quadrilateral inscribed in a circle. Prove that $\frac{AC}{BD} = \frac{AB\cdot AD+BC\cdot CD}{AB\cdot BC+AD\cdot CD}$.

Problem 17. Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let AC and BD meet at E, and let point F lie on segment BC such that EF is parallel to AB. Prove that $\frac{1}{EF} = \frac{1}{AB} + \frac{1}{CD}$.

Problem 18. Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?

Problem 19. Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45°$. What is $CE^2 + DE^2$?

3 Harder Problems

Problem 20. Quadrilateral $ABCD$ is inscribed in circle O and has sides $AB = 3$, $BC = 2$, $CD = 6$, and $DA = 8$. Let X and Y be points on $\overline{BD}$ such that

$$\frac{DX}{BD} = \frac{1}{4} \quad \text{and} \quad \frac{BY}{BD} = \frac{11}{36}.$$

Let E be the intersection of intersection of line AX and the line through Y parallel to $\overline{AD}$. Let F be the intersection of line CX and the line through E parallel to $\overline{AC}$. Let G be the point on circle O other than C that lies on line CX. What is $XF \cdot XG$?

Problem 21. (Desargues's Theorem). Let ABC be a triangle and let P be an interior point. Points A_1, B_1, C_1, respectively, lie on segments AP, BP, CP; assume that $A_1 \neq A, P; B_1 \neq B, P; C_1 \neq C, P$. Let lines B_1C_1 and BC meet at point A_2, lines A_1C_1 and AC meet at point B_2, and lines A_1B_1 and AB meet at point C_2. Prove that A_2, B_2, and C_2 are collinear.

Problem 22. In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Problem 23. Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z. Prove that

$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

Problem 24. (Pascal's Theorem). Let A, E, C, F, B, D be six (pairwise) distinct points lying on a circle in that order. Let segments AB and DE meet at X, let segments BC and EF meet at Y, and let segments CD and FA meet at Z. Prove that X, Y, and Z are collinear.

Problem 25. Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

Problem 26. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ passes through the midpoint of segment $\overline{EF}$.

Problem 27. Let ABC be a triangle, and let P be a point inside it. Let O, D, E, F be the circumcenters, respectively, of $\triangle ABC, \triangle BPC, \triangle CPA, \triangle APB$, and let T be the intersection of BC with EF. Let Q be the reflection of O over line EF. Prove that DQ is perpendicular to PT.