UT Arlington Mid-Cities Math Circle $(MC)^2$
Coloring Related Problems
September 9, 2020

Warm-up Problems

Problem 1. Two opposite corners of a 4×4 board are removed. Can the remaining figure be covered with dominoes of the shape 1×2?

Problem 2. Two opposite corners of an 8×8 board are removed. Can the remaining figure be covered with dominoes of the shape 1×2?

More Difficult Problems

Problem 3. A rectangular room has a floor tiled with tiles of two shapes: 2×2 and 1×4. The tiles completely cover the floor of the room, and no tile has been damaged, or cut in half. One day, a heavy object is dropped on the floor and one of the tiles is cracked. The handyman removes the damaged tile and goes to the storage to get a replacement. But he finds that there is only one spare tile, and it is of the other shape. Can he rearrange the remaining tiles in the room in such a way that the spare tile can be used to fill the hole?

Problem 4. Is there a way to pack 250 bricks of dimension $1 \times 1 \times 4$ into a $10 \times 10 \times 10$ box?

Problem 5. Is there a way to pack 53 bricks of dimension $1 \times 1 \times 4$ into a $6 \times 6 \times 6$ box? The faces of the bricks are parallel to the faces of the box.

Problem 6. A beetle sits on each square of a 9×9 chessboard. At a signal each beetle crawls diagonally onto a neighboring square. Then it may happen that several beetles will sit on some squares and none on others. Find the minimal possible number of free squares.

Problem 7. Given n points ($n \geq 5$) in the plane, prove that these n points can be colored by two colors so that no line can separate the points of one color from those of the other color.

Problem 8. Every point in the plane is colored in red or blue. The sets R and B consist of the lengths of those segments in the plane with both endpoints red and blue, respectively. Show that at least one of these sets contains all positive real numbers.

Problem 9. Every point in the space is colored with exactly one of the colors red, green, or blue. The sets R, G, B consist of the lengths of those segments in space with both endpoints red, green, and blue, respectively. Show that at least one of these sets contains all positive real numbers.

Problem 10. A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?

Problem 11. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Problem 12. (*The Art Gallery Problem*) An art gallery has the shape of a simple n-gon. Find the minimum number of watchmen needed to survey the building, no matter how complicated its shape.

Problem 13. Let $\mathcal{P}$ be a finite set of squares on an infinite chessboard. Kelvin the Frog notes that $\mathcal{P}$ may be tiled with only 1×2 dominoes, while Alex the Kat notes that $\mathcal{P}$ may be tiled with only 2×1 dominoes. The dominoes cannot be rotated in each tiling. Prove that the area of $\mathcal{P}$ is a multiple of 4.

Problem 14. Consider an $n \times n$ chessboard. Define a rook to be a 1×1 square which cannot be placed in the same row or same column as another rook. Define a domino to be a 2×1 rectangle which cannot overlap, either with other dominoes or other rooks. For which positive integer values of n is it possible to tile the chessboard completely with n rooks and the rest dominoes?

Problem 15.

(a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if three is replaced by nine?

Problem 16. A 23×23 square is completely tiled by 1×1, 2×2, and 3×3 tiles. What minimum number of 1×1 tiles are needed?

Problem 17. A configuration of 4039 points in the plane is called Texan if it consists of 2019 red points and 2020 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Texan configuration if the following two conditions are satisfied:

(a) No line passes through any point of the configuration.

(b) No region contains points of both colors.

Find the least value of k such that for any Texan configuration of 4039 points, there is a good arrangement of k lines.