INVARIANTS

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An invariant is some aspect of a problem (usually a numerical property) that does not change. Invariants can be used in many ways; a few of the many "flavors" of invariant problems are provided below. Generally, a solution which utilizes invariants should proceed as follows:

- Find a useful invariant–a quantity or property that does not change throughout the course of the problem (this can require some creativity!)
- Prove that it is actually invariant!
- Prove that the invariant implies the desired result.

Warm-Up Problems

(1) There are 35 people in a Zoom meeting. Each minute, either 6 people enter, or 10 people leave. The Zoom meeting cannot end until everyone has left. Is it possible that the meeting will ever end?

(2) Hercules is fighting a Hydra with 100 heads. He has a sword that can slice off exactly 21 of its heads, and a dagger that can slice off 4 of its heads. However, if he uses the dagger, the Hydra will grow 2020 additional heads. (Note: if the Hydra has 3 or fewer heads, neither of Hercules' weapons can be used.) Is it possible for Hercules to slay the Hydra?

(3) Write the numbers $1, 2, \ldots, 20$ on the board. At each step, select two numbers a and b, erase them, and write the number $a+b-1$ on the board. Do this until one number remains. What number is left at the end?

(4) Again write the numbers $1, 2, \ldots, 20$. This time, at each step, erase a and b and write $ab + a + b$. What number is left at the end?

(5) Write the number 8^{2020} on the board. At each step, replace the number with the sum of its digits, until a single digit is left. What number is left at the end?

(6) Let $a_1, a_2, \ldots a_n$ represent some reordering of $1, 2, \ldots n$. For example, if $n = 3$, we could have $a_1 = 2, a_2 = 3, a_3 = 1$. Prove that if n is odd, the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3) \ldots (a_n - n)$$

is even.

(7) If 75 people play in a tennis tournament, prove that at any point, the number of people who have played in an odd number of games is even.

(8) 7 coins are placed on a table. All of them are initially facing tails up. At each stage, you may flip any 4 of the coins. Is it possible to end up with all of the coins heads up?

(9) Boxes of an $m \times n$ grid are filled so that the sum of numbers in each row and in each column is equal to 1. Prove that $m = n$.

(10) Write a version of problem number (5) which uses the properties of the number 11.

(11) Chris and Emily are playing chess. To make things interesting, they decide to add a special piece called a “camel”, which can make a“1x3 knight move” (e.g., one space up, and three spaces right). How many moves are needed for a camel to end up in a square adjacent to its initial square?

(12) Students in a class are seated in an $n \times n$ grid. The teacher decides that he would like to rearrange the seating, so he asks each student to move seats. However, to prevent the class from falling into utter chaos, he tells the students that they can only move into an adjacent chair in the same row or same column (i.e., they can only move one chair horizontally or one chair vertically). Prove that this is always possible if n is even, and impossible if n is odd.