Functional Equations, at the UTA $(MC)^2$

Luke Robitaille

April 16, 2020

A comment about notation: Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^*$, $\mathbb{Z}$, and $\mathbb{Q}$ denote, respectively, the sets of real numbers, positive real numbers, nonzero real numbers, integers, and rational numbers. (Why is the letter Z used for the set of integers? In German, the word for "numbers" is *Zahlen*, which begins with the letter Z.)

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(xy) = xf(y).$$

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(x^2 + y^2 + x + y) - x^2 - y^2 = f(x) + y.$$

Problem 3.

a) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for any integers x and y, $f(x + y) = f(x) + f(y)$.

b) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that, for any rational numbers x and y, $f(x + y) = f(x) + f(y)$.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(3x^{2020} + 8y) + f(x) = f(x^{65} + 7y).$$

Problem 5. Find all functions $f : \mathbb{R}^* \to \mathbb{R}$ such that, for any nonzero real number x,

$$\frac{1}{x}f(-x) + f\left(\frac{1}{x}\right) = x.$$

Problem 6. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that, for any rational numbers $x < y < z < t$ that form an arithmetic progression,

$$f(x) + f(t) = f(y) + f(z).$$

Problem 7. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for any integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Problem 8. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any positive real number x, $f(x) \leq 1000x^2$ and $f(f(x)) = x^4$.

Problem 9. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any positive real numbers w, x, y, and z that satisfy the equation $wx = yz$,

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}.$$

Problem 10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(xy + f(x)) = xf(y).$$

Problem 11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(x^2 - y^2) = xf(x) - yf(y).$$

Problem 12. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that, for any integers x and y,

$$f(x-f(y)) = f(f(x)) - f(y) - 1.$$

Problem 13. Find all ordered pairs of integers (a,b) for which there exist functions $f\colon\mathbb{Z}\to\mathbb{Z}$ and $g\colon\mathbb{Z}\to\mathbb{Z}$ such that, for any integer x,

$$f(g(x)) = x+a \quad \text{and} \quad g(f(x)) = x+b.$$

Problem 14. Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ such that, for any positive real numbers x and y,

$$\left(x+\frac{1}{x}\right)f(y) = f(xy) + f\left(\frac{y}{x}\right).$$

Problem 15. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$