UT Arlington Mid-Cities Math Circle $(MC)^2$
Sequences

Problem 1. $a_0 = a_1 = 1, a_{n+1} = a_{n-1}a_n + 1, (n > 1)$. Show that 4 does not divide a_{2012}.

Problem 2. $a_1 = a_2 = 1, a_n = \frac{a_{n-1}^2+2}{a_{n-2}}, (n > 3)$. Show that all a_i are integers.

Problem 3. Prove that all terms of the sequence $a_1 = a_2 = a_3 = 1, a_{n+1} = \frac{1+a_{n-1}a_n}{a_{n-2}}$ are integers.

Problem 4. $a_1 = a_2 = 1, a_3 = -1, a_n = a_{n-1}a_{n-3}$. Find a_{2012}.

Problem 5. Find the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}.$$

Problem 6. Find the sum

$$\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \ldots + \frac{1}{n(n+1)(n+2)(n+3)}.$$

Problem 7. Let $a_0, a_1, ..., a_n$ be a sequence such that $a_0 = a_n = 0$ and $a_{k-1} - 2a_k + a_{k+1} \geq 0$ for all $k = 1, ..., n-1$. Prove that $a_k \leq 0$ for all k.

Problem 8. The sequence $a_0, a_1, a_2, ...$ is such that, for all nonnegative $m, n (m \geq n)$, we have $a_{m+n} + a_{m-n} = (a_{2m} + a_{2n})/2$. Find a_{2012} if $a_1 = 1$.