UT Arlington Mid-Cities Math Circle $(MC)^2$
Complex Numbers and Polynomials

Problem 1. Solve the equation $z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0$.

Problem 2. Solve the equation

$4z^{11}+4z^{10}-21z^9-21z^8+17z^7+17z^6+17z^5+17z^4-21z^3-21z^2+4z+4 = 0.$

Division of polynomials. For any polynomials $f(x)$ and $g(x)$ there exist polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x), \quad \deg r < \deg g \ \text{ or } r(x) = 0.$$

The coefficients of the polynomials can be in: $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$. For example, if $f(x) = x^7 - 1$ and $g(x) = x^3 + x + 1$ then the quotient $q(x)$ is $x^4 - x^2 - x + 1$ and the remainder $r(x)$ is $2x^2 - 2$. In the case $g(x) = x - a$ we obtain an important fact: $f(a) = 0$ if and only if $f(x) = (x - a)q(x)$ for some polynomial $q(x)$

Problem 3. Factor the following polynomials as products of irreducible polynomials with integer coffeicients.
(a) $x^4 + x^2 + 1$, (b) $x^{10} + x^5 + 1$, (c) $x^9 + x^4 - x - 1$.

Problem 4. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder of $f(x^5)$ when divided by $f(x)$.

Problem 5. Find the remainder of x^{1959} when divided by $(x^2+1)(x^2+x+1)$.

Problem 6. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all complex numbers z. Prove that $f(z)+f(-z) = 0$ for all complex numbers z.

Problem 7. Find all polynomials $f(x)$, for which $f(x)f(2x^2) = f(2x^3 + x)$.

Problem 8. Find all polynomials f, for which $f(x^2) + f(x)f(x + 1) = 0$.

Problem 9. (USAMO 1977) If a and b are two solutions of $x^4+x^3-1 = 0$, then ab is a solution of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

Problem 9. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all complex numbers z. Prove that $f(z)+f(-z) = 0$ for all complex numbers z.