**UT Arlington Mid-Cities Math Circle** $(MC)^2$
**Invariants**

**Problem 1.** Assume you have an $8 \times 8$ chessboard with the usual coloring. You may repaint all squares

(a) of a row or column.

(b) of a $2 \times 2$ square.

The goal is to attain just one black square. Can you reach the goal?

**Problem 2.** Suppose the positive integer $n$ is odd. First Al writes the numbers $1, 2, ..., 2n$ on the blackboard. Then he picks any two numbers $a, b$, erases them, and writes, instead, $|a - b|$. Prove that an odd number will remain at the end.

**Problem 3.** The integers $1, ..., 2n$ are arranged in any order on $2n$ places numbered $1, ..., 2n$. Now we add its place number to each integer. Prove that there are two among the sums which have the same remainder $\mod 2n$.

**Problem 4.** There are $a$ white, $b$ black, and $c$ red chips on a table. In one step, you may choose two chips of different colors and replace them by a chip of the third color. If just one chip will remain at the end, its color will not depend on the evolution of the game. For what $a, b, c$ can this final state be reached?

**Problem 5.** There are $a$ white, $b$ black, and $c$ red chips on a table. In one step, you may choose two chips of different colors and replace each one by a chip of the third color. Find conditions for all chips to become of the same color. Suppose you have initially 13 white 15 black and 17 red chips. Can all chips become of the same color? What states can be reached from these numbers?

**Problem 6.** There is a row of 1000 integers. There is a second row below, which is constructed as follows. Under each number $a$ of the first row, there is a positive integer $f(a)$ such that $f(a)$ equals the number of occurrences of $a$ in the first row. In the same way, we get the 3rd row from the 2nd row, and so on. Prove that, finally, one of the rows is identical to the next row.

**Problem 7.** There is an integer in each square of an $8 \times 8$ chessboard. In one move, you may choose any $4 \times 4$ or $3 \times 3$ square and add 1 to each integer of the chosen square. Can you always get a table with each entry divisible by

(a) 2,
(b) 3?

**Problem 8.** Suppose not all four integers $a, b, c, d$ are equal. Start with $(a, b, c, d)$ and repeatedly replace $(a, b, c, d)$ by $(a-b, b-c, c-d, d-a)$. Then at least one number of the quadruple will eventually become arbitrarily large.