UT Arlington Mid-Cities Math Circle $(MC)^2$
Symmetries on the Plane

1. * [William Lowell Putnam Mathematics Competition, 1990]
   Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

2. Can you find a convex geometric figure, which has no axis of symmetry, but is invariant (i.e. stays the same) under some rotation.

3. * Give an example of a polynomial $P(x)$ of degree exactly 2011, for which it is known that $P(x) + P(1 - x) \equiv 1$.

4. Is there a bounded geometric figure, which has two parallel axes of symmetry? How about an unbounded figure?

5. How many axes of symmetry can a triangle have?

6. * The numbers $1, 2, \ldots, 2011$ are placed in the vertices of a regular 2011-gon (a polygon with 2011 vertices). Each axis of symmetry splits the numbers (that are not on it) to two sets. We will call the placement "good" with respect to the given axis, if any number of one set is larger than its symmetric one in the other set. Is there a placement, which is "good" with respect to all axes of symmetry?

7. Is it true, that if a quadrilateral has an axis of symmetry, then it is either a rectangle, a rhombus or an isosceles trapezoid?

8. * Let f be a given function defined on the whole real line. Find two functions f_1 and f_2 such that $f = f_1 + f_2$, the graph of f_1 is symmetric with respect to line $x = 0$, and the graph of f_2 is symmetric with respect to line $x = 1$.