UT Arlington Mid-Cities Math Circle $(MC)^2$
Inversion

An *inversion* $I(O, r)$ is defined by its *center* O and its radius r, or by the circle $k = k(O, r)$. A point A goes to a point A_1 under the inversion $I(O, r)$ if A_1 is on the ray $\overrightarrow{OA}$ and $OA_1 = \frac{r^2}{OA}$. We will write $I(A) = A_1$. We have three separate cases depending on the position of A: (i) A is inside the circle k; (ii) A is on the circle k; (iii) A is outside the circle k.

Problem 1. Prove that if $I(A) = A_1$ then $I(A_1) = A$.

Problem 2. Find the images under $I(O, r)$ of:
(a) a line l that passes through the center O, i.e. $I(l) =?$;
(b) a line l that does not pass through the center O;
(c) a circle k_1 that passes through the center O, i.e. $I(k_1) =?$;
(d) a circle k_1 that does not pass through the center O;

Problem 3. Let k_1 and k_2 be two tangent to each other circles, both of them passing through O. What can we say about (the lines) $I(k_1)$ and $I(k_2)$? Are there any other interesting relations between a pair of objects and their immages?

Problem 4. The circles k_1, k_2, k_3, and k_4 are positioned in such a way that k_1 is tangent to k_2 at a point A, k_2 is tangent to k_3 at a point B, k_3 is tangent to k_4 at a point C, and k_4 is tangent to k_1 at a point D. Show that A, B, C, and D are collinear (i.e. on the same line) or concyclic (i.e. on the same circle).

Problem 5. The circles k_1, k_2, k_3, and k_4 intersect cyclicly in the following pairs of points: A_1, A_2; B_1, B_2; C_1, C_2; and D_1, D_2 (i.e. k_1 and k_2 intersect at A_1 and A_2, etc.). Prove the following
(a) If A_1, B_1, C_1, D_1 are collinear/concyclic, then so are A_2, B_2, C_2, D_2.
(a) If A_1, A_2, C_1, C_2 are collinear/concyclic, then so are B_1, B_2, D_1, D_2.

Problem 6. If A and B are distinct points different from O, with $I(A) = A_1$ and $I(B) = B_1$, then prove that ΔOAB is similar to ΔOB_1A_1 and that $A_1B_1 = \frac{AB \cdot r^2}{OA \cdot OB}$.

Problem 7. (Ptolemy's Theorem) A quadrilateral $ABCD$ is inscribed in a circle k. Prove that $AB \cdot CD + AD \cdot BC = AC \cdot BD$.