An inversion $I(O, r)$ is defined by its center $O$ and its radius $r$, or by the circle $k = k(O, r)$. A point $A$ goes to a point $A_1$ under the inversion $I(O, r)$ if $A_1$ is on the ray $\overrightarrow{OA}$ and $OA_1 = \frac{r^2}{OA}$. We will write $I(A) = A_1$. We have three separate cases depending on the position of $A$: (i) $A$ is inside the circle $k$; (ii) $A$ is on the circle $k$; (iii) $A$ is outside the circle $k$.

**Problem 1.** Prove that if $I(A) = A_1$ then $I(A_1) = A$.

**Problem 2.** Find the images under $I(O, r)$ of:
   (a) a line $l$ that passes through the center $O$, i.e. $I(l) =$?;
   (b) a line $l$ that does not pass through the center $O$;
   (c) a circle $k_1$ that passes through the center $O$, i.e. $I(k_1) =$?;
   (d) a circle $k_1$ that does not pass through the center $O$;

**Problem 3.** Let $k_1$ and $k_2$ be two tangent to each other circles, both of them passing through $O$. What can we say about (the lines) $I(k_1)$ and $I(k_2)$? Are there any other interesting relations between a pair of objects and their images?

**Problem 4.** The circles $k_1, k_2, k_3,$ and $k_4$ are positioned in such a way that $k_1$ is tangent to $k_2$ at a point $A$, $k_2$ is tangent to $k_3$ at a point $B$, $k_3$ is tangent to $k_4$ at a point $C$, and $k_4$ is tangent to $k_1$ at a point $D$. Show that $A, B, C,$ and $D$ are collinear (i.e. on the same line) or concyclic (i.e. on the same circle).

**Problem 5.** The circles $k_1, k_2, k_3,$ and $k_4$ intersect cyclically in the following pairs of points: $A_1, A_2; B_1, B_2; C_1, C_2; \text{ and } D_1, D_2$ (i.e. $k_1$ and $k_2$ intersect at $A_1$ and $A_2$, etc.). Prove the following
   (a) If $A_1, B_1, C_1, D_1$ are collinear/concyclic, then so are $A_2, B_2, C_2, D_2$.
   (a) If $A_1, A_2, C_1, C_2$ are collinear/concyclic, then so are $B_1, B_2, D_1, D_2$.

**Problem 6.** If $A$ and $B$ are distinct points different from $O$, with $I(A) = A_1$ and $I(B) = B_1$, then prove that $\Delta OAB$ is similar to $\Delta OB_1A_1$ and that $A_1B_1 = \frac{AB \cdot r^2}{OA \cdot OB}$.

**Problem 7.** (Ptolemy’s Theorem) A quadrilateral $ABCD$ is inscribed in a circle $k$. Prove that $AB \cdot CD + AD \cdot BC = AC \cdot BD$. 