UT Arlington Mid-Cities Math Circle $(MC)^2$
Number Theory: Divisibility and Remainders

"Mathematics is the queen of the sciences and Number Theory is the queen of mathematics" –Gauss

Let a and b be integers and d be a positive integer. We say that *a is congruent to b modulo d* if d divides $a - b$. We write $a \equiv b \pmod{d}$.

Problem 1. Let $a \equiv b \pmod{d}$ and $c \equiv e \pmod{d}$. Prove that
(a) $a + c \equiv b + e \pmod{d}$;
(b) $a - c \equiv b - e \pmod{d}$;
(c) $ac \equiv be \pmod{d}$.

Problem 2. Find the remainder of 2^{2010} when divided by 5. What if divided by 7? And what if divided by 35?

Problem 3. Calculate the last digits of 2^{2010}, 3^{2010}, and 7^{2010}.

Fermat's Little Theorem: If p is a prime number and a is an integer not divisible by p then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Euler's Theorem: If a and n are relatively prime positive integers and $\phi(n)$ is the number of integers between 1 and n that are relatively prime to n then

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Problem 4. Show that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Problem 5. Prove that $1^n + 2^n + ... + (n-1)^n$ is divisible by n for any odd $n > 1$.

Problem 6. Find all integers x and y for which $x^2 - 3y^2 = 17$.

Problem 7. Prove that no three integers x, y, z satisfy

$$x^3 + y^3 + z^3 = 500$$

Problem 8. (USAMO 1998) Suppose that the set $\{1, 2, ..., 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \le i \le 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum

$$|a_1 - b_1| + |a_2 - b_2| + ... + |a_{999} - b_{999}|$$

ends in the digit 9.