UT Arlington Mid-Cities Math Circle $(MC)^2$
Algebra Problems

Problem 1. (AMC10, 2005) Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is $abcd$?

Problem 2. (AMC10, 2005)For each positive integer $m > 1$, let $P(m)$ denote the greatest prime factor of m. For how many positive integers n is it true that both $P(n) = \sqrt{n}$ and $P(n+48) = \sqrt{n+48}$?

Problem 3. (AIME1, 2003) An integer between 1000 and 9999, inclusive, is called *balanced* if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there?

Problem 4. (AIME2, 2005) Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$$

Find $(x+1)^{48}$.

Problem 5. (AIME2, 2003) Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Problem 6. (USAMO, 2003) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8$$

Problem 7. (USAMO, 2001) Let a, b, and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2$$